

Exponentially decaying eigenvectors for certain almost periodic operators

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Abstract. For every point χ in the spectrum of the operator

$$(h(\delta)\xi) = \xi_{n+1} + \xi_{n-1} + \beta (\delta e^{2\pi\alpha ni} + \delta^{-1} e^{-2\pi\alpha ni}) \xi_n$$

on $\ell^2(\mathbb{Z})$ there exists a complex number x of modulus one such that the equation

$$\xi_{n+1} + \xi_{n-1} + \beta (x\delta e^{2\pi\alpha ni} + \bar{x}\delta^{-1} e^{-2\pi\alpha ni}) \xi_n = \chi \xi_n$$

has a non-trivial solution satisfying the condition

$$\overline{\lim}_{|n| \rightarrow \infty} |\xi_n|^{1/|n|} \leq \delta^{-1} \beta^{-1}$$

provided that $\beta, \delta > 1$ and α satisfies the diophantine condition

$$\lim_{n \rightarrow \infty} |\sin \pi \alpha n|^{-\frac{1}{n}} = 1.$$

The parameters $x\delta$ and χ are in the range of analytic functions which are defined on a Riemann surface covering the resolvent set of the operator $h(1)$.

Introduction

The spectral properties of almost periodic operators have been investigated extensively over the past 25 years, both in mathematics as well as in physics. A particularly intriguing problem in this area this author has been occupied with for some time is the question under what conditions such operators have point spectrum and how the prevalence of point spectrum affects the topological nature of their spectrum. In certain cases the second part of this question appears to be intimately related to a problem that deserves some consideration in its own right, namely, simplistically put, under what conditions is every point in the spectrum an eigenvalue?

In the sequel a family of almost periodic operators will be considered for which this question has a satisfactory answer. The operators to be considered are complex perturbations of bounded self-adjoint operators, known as almost Mathieu operators or Harper's

operators. The approach chosen is C^* -algebraic. The key to the proofs are certain automorphisms ρ_β of the irrational rotation C^* -algebra \mathcal{A}_α associated with an irrational number α . If u and v are unitary generators of \mathcal{A}_α satisfying the defining relation $uv = e^{2\pi\alpha i}vu$, the operators of interest are of the form

$$h(\delta) = u + u^* + \beta(\delta v + \delta^{-1}v^*)$$

where $\beta > 1$ is a fixed constant and $|\delta| > 1$. The said automorphism ρ_β flips a defining parameter when applied to a slightly enlarged family of operators. This can then be used to generate exponentially decaying eigenvectors for the operators $h(\delta)$ represented on the Hilbert space $\ell^2(\mathbb{Z})$, provided α satisfies a suitable diophantine condition. From a dynamical systems point of view, the automorphism ρ_β is related to a skew translation, extending the irrational rotation underlying the dynamics of the operators in question, in a sense to be made precise below. It is noteworthy that as β approaches 1, ρ_β approaches an automorphism of period 4, a so-called ‘‘Fourier transform’’. This shows, in particular, that the extension of the dynamical systems picture, which is so vital for the case $\beta > 1$, is no longer available in the case $\beta = 1$.

Introducing the automorphism ρ_β and presenting a brief discussion of the extended dynamical systems picture will be taken up in the first paragraph. In the second paragraph the existence of exponentially decaying eigenvectors for points in the spectrum of the operators $h(\delta)$ will be proved. It will then be shown that a far more specific formulation of the eigenvalue problem for the operators $h(\delta)$ can be obtained through parametrization in a suitable Riemann surface covering the resolvent set \mathcal{R} of the operator $h(1)$. More specifically, since the spectrum of $h(1)$ turns out to be a regular compactum in the sense of potential theory, and since the spectra of the operators $h(\delta)$ are exactly the level curves of the corresponding conductor potential, there exists a Riemann surface $\tilde{\mathcal{R}}$ covering \mathcal{R} and an analytic function G which maps $\tilde{\mathcal{R}}$ onto the complement of the closed unit disk, such that

$$h(G(z))\xi = p(z)\xi$$

has an exponentially decaying solution ξ for every z in $\tilde{\mathcal{R}}$. Here p denotes the canonical mapping from $\tilde{\mathcal{R}}$ onto \mathcal{R} . As z ranges over $\tilde{\mathcal{R}}$, ξ ranges over all possible eigenvectors for the operators $h(\delta)$. Due to the basic K -theory for the C^* -algebra \mathcal{A}_α , one can see that the group of covering transformations of $\tilde{\mathcal{R}}$ over \mathcal{R} is infinite cyclic. Moreover, translation by one of the two generators of this group, ω say, corresponds to shifting the eigenvector ξ .

The automorphism ρ_β gives rise to an eigenvalue problem in its own right which is intimately interconnected with the one expounded above. The eigenvalues are given

through an analytic function Γ on $\tilde{\mathcal{R}}$ which has the property

$$\Gamma(\omega(z)) = G(z)^2 \Gamma(z) .$$

In paragraph 3 it will be shown that the two eigenvalue problems are essentially equivalent. To this end, the latter eigenvalue problem will be transformed into a question regarding the kernel of certain Fredholm operators of index zero. The problem then boils down to the question whether these kernels are one-dimensional. With the aid of analytic perturbation theory, it will be shown that this is indeed the case.

Finally, in paragraph 4, the case $|\delta| = 1$ will be discussed. Since $h = h(1)$ is a fixed-point of ρ_β , no exponentially decaying eigenvectors can be generated along the lines this was possible for the case $|\delta| > 1$. Nevertheless, the extended dynamical systems picture shows that the eigenvalue problem for h is intimately related to similar questions about Schrödinger type difference operators with unbounded potentials, such as

$$n \mapsto \tan \pi(\alpha n^2 + 2\theta n + \nu) .$$

Even though these operators are distinctly non almost periodic, they are related to, and in fact extensions of, a family of operators which was a focus of research in the early 1980's (let $\alpha = 0$ and let θ be irrational). In the restricted case ($\alpha = 0$) it can be shown that the said operators have pure point spectrum. This is being accomplished by relating these operators to certain bounded operators which can be diagonalized by solving a specific cocycle equation, and then by translating this information back to the original context. In the extended case ($\alpha \neq 0$) there still exist those related bounded operators, which are actually derived from the automorphism ρ_β , and which are the point of departure, rather than an auxiliary device, as it happens to be the case when $\alpha = 0$. But it is not possible anymore to diagonalize these related bounded operators, due to the complications brought about by switching from an irrational rotation to a skew translation extending it.

1. The automorphism ρ_β

For an irrational number α we consider the C^* -algebra $\mathcal{A} = \mathcal{A}_\alpha$ generated by two unitaries u and v satisfying the relation $uv = \lambda^2 vu$, where $\lambda = e^{\pi\alpha i}$. For $\beta \neq 1$, let

$$\begin{aligned} \rho_\beta(u) &= vuv(uv + \beta)^{-1}(v^*u^* + \beta) \\ \rho_\beta(v) &= v(uv + \beta)^{-1}(v^*u^* + \beta) . \end{aligned}$$

Then $\rho_\beta(u)$ and $\rho_\beta(v)$ are unitaries satisfying again the relation

$$\rho_\beta(u)\rho_\beta(v) = \lambda^2 \rho_\beta(v)\rho_\beta(u) .$$

Therefore, ρ_β extends to an automorphism of \mathcal{A} which we will also denote by ρ_β . The significance of this automorphism for what is to follow rests with the identities

$$(1.1) \quad \rho_\beta(u + \beta v) = u^* + \beta v \quad , \quad \rho_\beta(u^* + \beta v^*) = u + \beta v^* \quad .$$

Let $GL(2, \mathbb{Z})$ be the group of 2×2 matrices with integer entries and a determinant of modulus 1. The assignment

$$\left. \begin{aligned} w_{m,n} &\rightarrow w_{pq} \\ \begin{pmatrix} p \\ q \end{pmatrix} &= A \begin{pmatrix} m \\ n \end{pmatrix} \end{aligned} \right\} \quad A \in GL(2, \mathbb{Z}) \quad ,$$

where $w_{pq} = \lambda^{-pq} u^p v^q$, is known to define a linear isometry of \mathcal{A} . This isometry is an automorphism if $\det A = 1$, and it is an antiautomorphism if $\det A = -1$. The mapping assigning to every $A \in GL(2, \mathbb{Z})$ the corresponding isometry is a faithful homomorphism from the group $GL(2, \mathbb{Z})$ into the group of isometries of \mathcal{A} . For convenience we will denote matrices in $GL(2, \mathbb{Z})$ and their corresponding isometries by the same symbol. For instance $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents the antiautomorphism $u \mapsto v$, $v \mapsto u$.

Returning to the automorphism ρ_β we are now going to list a number of useful identities.

$$(1.2) \quad \left. \begin{aligned} \rho_\beta &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \circ \rho_\beta^{(0)} \circ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad , \text{ where } \\ \rho_\beta^{(0)}(u) &= u \\ \rho_\beta^{(0)}(v) &= v(\lambda u + \beta)^{-1}(\bar{\lambda} u^* + \beta) \end{aligned} \right\}$$

$$(1.3) \quad \rho_{\beta^{-1}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \circ \rho_\beta^{-1} \circ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

This identity says essentially that $\rho_{\beta^{-1}}$ and $\rho_\beta^{-1} \circ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ are conjugates of each other via a ‘‘Fourier transform’’ (also known as ‘‘duality’’).

$$(1.4) \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \circ \rho_\beta = \rho_\beta \circ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

If β approaches 1 then $\rho_\beta^{(0)}$ approaches $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ on w_{pq} , hence ρ_β approaches

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

on w_{pq} . Thus ρ_β approaches a “Fourier transform”. Throughout in the subsequent discussion we will limit our attention to irrational numbers α only which satisfy a diophantine condition.

$$(1.5) \quad \lim_{n \rightarrow \infty} |\sin \pi \alpha n|^{-1} = 1$$

For such numbers α and $\beta > 1$ the automorphism $\rho_\beta^{(0)}$ becomes an inner automorphism. Namely,

$$(1.6) \quad \left\{ \begin{array}{l} \rho_\beta^{(0)}(a) = e^{ig(u)} a e^{-ig(u)}, a \in \mathcal{A} ; \text{ or } \rho_\beta^{(0)} = \text{Ad}(e^{ig(u)}) , \\ \text{where} \\ g(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (\sin \pi \alpha n)^{-1} \beta^{-n} (z^n + z^{-n}) . \end{array} \right.$$

We turn now to a dynamical systems interpretation of the automorphisms ρ_β in the framework of C^* -algebras, under the assumption that (1.5) holds. Consider the crossed product \mathcal{B} of \mathcal{A} by the automorphism $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$$\mathcal{B} = \mathcal{A} \otimes \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \mathbb{Z} .$$

This C^* -algebra is generated by three unitaries u , v and w satisfying the defining relations

$$\left. \begin{array}{l} w^* u w = \bar{\lambda} u v \\ v w = w v \\ u v = \lambda^2 v u \end{array} \right\}$$

Since $\rho_\beta = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \circ \text{Ad}(e^{ig(u)}) \circ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, ρ_β extends to an inner automorphism of \mathcal{B} . So \mathcal{B} provides a natural framework where all our manipulations so far take place. Manipulating the first of those relations we get

$$u w u^* = \lambda v w .$$

This, in conjunction with the other two relations, suggests that \mathcal{B} can also be realized as the crossed-product of the C^* -algebra of continuous functions on the two dimensional torus $C(\mathbb{T}^2)$ by a skew translation followed by a translation. More specifically, let

$$\phi(x, y) = \lambda(\lambda x, xy) , \quad x, y \in \mathbb{T}^2 .$$

In dynamical systems theory it is a well known fact that ϕ is a uniquely ergodic homeomorphism of \mathbb{T}^2 , the unique invariant probability measure being the Haar measure on

the compact group \mathbb{T}^2 , which has of course full support. It is then a well known fact in C^* -algebra theory, that the crossed-product

$$\tilde{\mathcal{B}} = C(\mathbb{T}^2) \otimes_{\phi} \mathbb{Z}$$

is a simple C^* -algebra (i.e., it has no non-trivial ideals) with a unique tracial state extending the Haar measure on $C(\mathbb{T}^2)$. Letting \tilde{u} be the unitary in $\tilde{\mathcal{B}}$ corresponding to ϕ , \tilde{v} the projection from \mathbb{T}^2 onto the first component, and \tilde{w} the projection from \mathbb{T}^2 onto the second component, then it is easily seen that \tilde{u} , \tilde{v} and \tilde{w} satisfy the same three relations stated above for u , v and w . Using this information it is not difficult to see that the assignments $\tilde{u} \mapsto u$, $\tilde{v} \mapsto v$, $\tilde{w} \mapsto w$ extend to an isomorphism from $\tilde{\mathcal{B}}$ onto \mathcal{B} .

2. Spectrum and point spectrum for a family of non self-adjoint almost periodic operators

In this paragraph we will employ the automorphism ρ_β to investigate the spectrum of the operators

$$h(\delta) = u + u^* + \beta(\delta v + \delta^{-1}v^*)$$

for $|\delta| > 1$, provided that $\beta > 1$ and α satisfies the property (1.5). To this end we consider the extended family

$$h_\gamma(\delta) = \gamma u + \gamma^{-1}u^* + \beta(\delta v + \delta^{-1}v^*) ,$$

where $\beta^{-1}\delta^{-1} < |\gamma| < \beta\delta$.

The assignments

$$(2.1) \quad \begin{cases} (u\xi)_n = \xi_{n-1} \\ (v\xi)_n = \bar{\lambda}^{2n}\xi_n \\ (w\xi)_n = \lambda^{n^2}\xi_n \end{cases}$$

define linear operators on the vector space \mathbb{C}^∞ of (two-sided) sequences ξ . When restricted to square summable sequences, these assignments extend to a (faithful) representation of the C^* -algebra \mathcal{B} on the Hilbert space $\ell^2(\mathbb{Z})$ introduced in paragraph 1. For $\beta^{-1} < |\gamma| < \beta$, let

$$k_\gamma = w e^{ig(\gamma u)} w , \quad k = k_1 .$$

Then (1.1), (1.2) and (1.6) yield

$$(2.2) \quad h_{\gamma\delta^{-1}}(\delta)k_\gamma = k_\gamma h_{\gamma\delta}(\delta) .$$

For any complex number x , let

$$(D_x\xi)_n = x^n\xi , \quad \xi \in \mathbb{C}^\infty .$$

2.1 Lemma. Let $|\delta| > 1$; $\beta^{-1} < |\gamma| < 1$ or $1 < |\gamma| < \beta$.

(+) Suppose that

$$h_{\gamma\delta}(\delta)\eta = z\eta \text{ for some } z \in \mathbb{C} , \eta \in \ell^\infty(\mathbb{Z}) .$$

Then

$$h(\delta)\xi = z\xi , \quad \overline{\lim}_{n \rightarrow -\infty} |\xi_n|^{1/|n|} \leq |\gamma|\delta^{-1} , \quad \overline{\lim}_{n \rightarrow \infty} |\xi_n|^{1/n} \leq |\gamma|^{-1}\delta^{-1}$$

where $\xi = D_{\gamma^{-1}\delta^{-1}}\eta$.

(-) Suppose that

$$h_{\gamma\delta^{-1}}(\delta)\eta = z\eta \quad \text{for some } z \in \mathbb{C}, \quad \eta \in \ell^\infty(\mathbb{Z}).$$

Then

$$h(\delta)\xi = z\xi, \quad \overline{\lim}_{n \rightarrow -\infty} |\xi_n|^{1/|n|} \leq |\gamma|^{-1}\delta^{-1}, \quad \overline{\lim}_{n \rightarrow \infty} |\xi_n|^{1/n} \leq |\gamma|\delta^{-1},$$

where $\xi = D_{\gamma^{-1}\delta}\eta$.

Proof: We will deal with the case (+) only. The case (-) can be handled in a similar fashion.

The identity (2.2) yields

$$h_{\gamma\delta^{-1}}k_\gamma\eta = zk_\gamma\eta, \quad \text{where } k_\gamma\eta \in \ell^\infty(\mathbb{Z}).$$

Let $\tilde{\xi} = D_{\gamma^{-1}\delta}\eta$. Then

$$h(\delta)\tilde{\xi} = z\tilde{\xi}, \quad \overline{\lim}_{n \rightarrow -\infty} |\tilde{\xi}_n|^{1/|n|} \leq |\gamma|\delta^{-1}, \quad \overline{\lim}_{n \rightarrow \infty} |\tilde{\xi}_n|^{1/n} \leq |\gamma|^{-1}\delta.$$

On the other hand,

$$h(\delta)\xi = z\xi, \quad \overline{\lim}_{n \rightarrow -\infty} |\xi_n|^{1/|n|} \leq |\gamma|\delta, \quad \overline{\lim}_{n \rightarrow \infty} |\xi_n|^{1/n} \leq |\gamma|^{-1}\delta^{-1}.$$

Therefore

$$\overline{\lim}_{n \rightarrow -\infty} |\xi_n \tilde{\xi}_{n+1}|^{1/|n|} \leq |\gamma|^2, \quad \overline{\lim}_{n \rightarrow \infty} |\xi_n \tilde{\xi}_{n+1}| \leq |\gamma|^{-2}$$

and

$$\overline{\lim}_{n \rightarrow -\infty} |\xi_{n+1} \tilde{\xi}_n|^{1/|n|} \leq |\gamma|^2, \quad \overline{\lim}_{n \rightarrow \infty} |\xi_{n+1} \tilde{\xi}_n| \leq |\gamma|^{-2}.$$

Since $|\gamma| \neq 1$ by assumption, this entails

$$\overline{\lim}_{n \rightarrow -\infty} (\xi_n \tilde{\xi}_{n+1} - \xi_{n+1} \tilde{\xi}_n) = 0$$

or

$$\overline{\lim}_{n \rightarrow \infty} (\xi_n \tilde{\xi}_{n+1} - \xi_{n+1} \tilde{\xi}_n) = 0.$$

Either way, it follows that ξ and $\tilde{\xi}$ are linearly dependent. For, if this were not the case, the expression following the $\overline{\lim}$ in the last two identities would have to be constant and non-zero for all $n \in \mathbb{Z}$. Therefore, observing that $\delta > 1$ by assumption, we conclude

$$\lim_{n \rightarrow -\infty} |\xi_n|^{1/|n|} \leq |\gamma|\delta^{-1}, \quad \lim_{n \rightarrow \infty} |\xi_n|^{1/n} \leq |\gamma|^{-1}\delta^{-1},$$

as claimed. ■

2.2 Lemma. Let \mathcal{I} be a set of real numbers and let $b > 0$, $d > 0$. Suppose that \mathcal{I} has a non-empty intersection with at least one of the two open intervals $(d - b, d)$ or $(d, d + b)$. Suppose in addition that \mathcal{I} has the following properties:

(+) $0 < |t| < b$ and $t + d \in \mathcal{I}$ implies $(t - d, t + d) \subset \mathcal{I}$.

(-) $0 < |t| < b$ and $t - d \in \mathcal{I}$ implies $(t - d, t + d) \subset \mathcal{I}$.

Then $(-d - b, d + b) \subset \mathcal{I}$.

Proof: Consider the case that $\mathcal{I} \cap (d - b, d) \neq \emptyset$. Then there exists a $c_0 \in (-b, 0)$ such that $c_0 + d \in \mathcal{I}$.

Suppose first that $b \leq d$.

Let $s > -d - b$, but close to $-d - b$. Since (+) ensures that $(c_0 - d, c_0 + d) \subset \mathcal{I}$, we can find $c_1 \in (-b, 0)$ such that $c_1 + d \in \mathcal{I}$ and $c_1 - d < s$. Again, (+) ensures that $(c_1 - d, c_1 + d) \subset \mathcal{I}$. Now let $t < d + b$ but close to $d + b$. Then we can find $c_2 \in (0, b)$ such that $c_2 - d \in (c_1 - d, c_1 + d) \subset \mathcal{I}$ and $c_1 + d > t$. Then (-) ensures that $(c_2 - d, c_2 + d) \subset \mathcal{I}$. By construction $(s, t) \subset (c_1 - d, c_1 + d) \cup (c_2 - d, c_2 + d) \subset \mathcal{I}$. Since s and t can be chosen arbitrarily close to $-d - b$ and $d + b$, respectively, we conclude that $(-d - b, d + b) \subset \mathcal{I}$.

Now suppose that $b > d$.

Let $s > -d - b$, but close to $-d - b$. Through induction we can generate a (possibly empty) chain $c_1 > \dots > c_n$ such that $c_0 > c_1$ if $n > 0$; $c_k \in (-b, 0)$, $c_k \in (c_{k-1} - d, c_{k-1} + d)$, for $1 \leq k \leq n$, and $c_n - d < -b$. Repeated applications of (+) show that $(c_k - d, c_k + d) \subset \mathcal{I}$ for $k = 0, \dots, n$. Now choose $c_{n+1} \in (-b, c_n)$ such that $c_{n+1} - d < s$. Once again, (+) ensures that $(c_{n+1} - d, c_{n+1} + d) \subset \mathcal{I}$. By construction we have $s \in (c_{n+1} - d, c_0 + d) \subset \mathcal{I}$.

Next, let $t < d + b$ but close to $d + b$. Using (-) instead of (+), we can construct in the same fashion a chain $c_0 < \tilde{c}_1 < \dots < \tilde{c}_{m+1}$ such that $t \in (c_0 - d, c_{m+1} + d) \subset \mathcal{I}$. This means $(s, t) \subset \mathcal{I}$, and again we conclude that $(-d - b, d + b) \subset \mathcal{I}$.

The case that $\mathcal{I} \cap (d, d + b) \neq \emptyset$ can be handled in a similar fashion. ■

2.3 Lemma. Let $|\delta| > 1$; $\beta^{-1} < |\gamma_0| < 1$ or $1 < |\gamma_0| < \beta$. Suppose that

$$h_{\gamma_0 \delta^{\pm 1}}(\delta)\eta = z\eta \text{ for some } z \in \mathbb{C}, \eta \in \ell^\infty(\mathbb{Z}).$$

Then

$$h(\delta)\xi = z\xi, \quad \overline{\lim}_{|n| \rightarrow \infty} |\xi_n|^{1/|n|} \leq \beta^{-1}|\delta|^{-1},$$

where $\xi = D_{\gamma_0^{-1}\delta_{\mp 1}}\eta$.

Proof: Let

$$b = \log \beta, \quad d = \log |\delta|, \quad c_0 = \log |\gamma_0|, \\ \mathcal{I} = \{ \log |\gamma| \mid D_\gamma \xi \in \ell^\infty(\mathbb{Z}) \}.$$

Then (+) and (−) in Lemma 2.1 translate into the namesake properties of Lemma 2.2, which then yields the desired conclusion. \blacksquare

Let $Sp_0(h_\gamma(\delta))$ be the spectrum of $h_\gamma(\delta)$ considered as a bounded linear operator on the Banach space $c_0(\mathbb{Z})$ of bounded two-sided sequences which vanish at infinity.

2.4 Lemma. For every $z \in Sp_0(h_\gamma(\delta))$ there exists $x \in \mathbb{T}$ and $\eta \in \ell^\infty(\mathbb{Z}) \setminus \{0\}$ such that

$$h_\gamma(x\delta)\xi = z\xi.$$

Proof: We choose an approximate eigenvector for z in $c_0(\mathbb{Z})$, $\eta^{(1)}, \eta^{(2)}, \dots$ in $c_0(\mathbb{Z})$, $\|\eta^{(m)}\|_\infty = 1$,

$$\lim_{m \rightarrow \infty} \|(h_\gamma(\delta) - z)\eta^{(m)}\|_\infty = 0.$$

For each m there is a j_m such that $|\eta_{j_m}^{(m)}| \geq \frac{1}{2}$. Let $\xi^{(m)} = u^{-j_m}\eta^{(m)}$. Then

$$\lim_{m \rightarrow \infty} \|(h_\gamma(x_m\delta) - z)\xi^{(m)}\|_\infty = 0,$$

where $x_m = \bar{\lambda}^{2j_m}$. Also,

$$|\xi_0^{(m)}| \geq \frac{1}{2}.$$

Since the unit ball of $\ell^\infty(\mathbb{Z})$ is weakly compact and metrizable, there exists a subsequence of $\xi^{(1)}, \xi^{(2)}, \dots$ which converges weakly to some $\xi \in \ell^\infty(\mathbb{Z})$. Moreover, we can arrange for the corresponding subsequence of x_1, x_2, \dots to converge to some $x \in \mathbb{T}$. Since $|\xi_0| \geq \frac{1}{2}$, ξ is non-zero. By construction

$$h_\gamma(x\delta)\xi = z\xi \quad \blacksquare$$

2.5 Lemma. $Sp_0(h_\gamma(\delta)) = Sp_0(h_{|\gamma|}(|\delta|))$.

Proof: Since $h_\gamma(\delta) = D_{\gamma/|\gamma|}h_{|\gamma|}(\delta)D_{\gamma/|\gamma|}^{-1}$, we have

$$Sp_0(h_\gamma(\delta)) = Sp_0(h_{|\gamma|}(\delta)).$$

Now let $z \in Sp_0(h_\gamma(\delta))$ and $x \in \mathbb{T}$. Then there exists an approximate eigenvector for z in $c_0(\mathbb{Z})$:

$$\eta^{(1)}, \eta^{(2)}, \dots; \quad \|\eta^{(m)}\|_\infty = 1 ,$$

$$\lim_{m \rightarrow \infty} \|(h_\gamma(\delta) - z)\eta^{(m)}\|_\infty = 0 .$$

For each m choose $j_m \in \mathbb{Z}$ such that $\lambda^{2j_m} \rightarrow x$. Then $u^{j_1}\eta^{(1)}, u^{j_2}\eta^{(2)}, \dots$ is seen to be an approximate eigenvector of $h_\gamma(x\delta)$ for z , which entails $z \in Sp_0(h_\gamma(x\delta))$. ■

2.6 Lemma. Let $|\delta| > 1$, $|\delta|\beta^{-1} < |\gamma| < |\delta|\beta$. Then

$$Sp_0(h_\gamma(\delta)) = Sp_0(h_{\gamma^{-1}}(\delta)) .$$

Proof: Let $z \in Sp_0(h_\gamma(\delta))$. By Lemma 2.4 there exist $x \in \mathbb{T}$, $\xi \in \ell^\infty(\mathbb{Z}) \setminus \{0\}$ such that

$$h_\gamma(x\delta)\xi = z\xi .$$

Lemma 2.3 implies that ξ and $\eta = D_{\gamma^{-2}}\xi$ are in $c_0(\mathbb{Z})$. But

$$h_{\gamma^{-1}}(x\delta)\eta = z\eta .$$

Hence

$$z \in Sp_0(h_{\gamma^{-1}}(x\delta)) ,$$

which, by Lemma 2.5, entails

$$z \in Sp_0(h_{\gamma^{-1}}(\delta)) .$$

so we have

$$Sp_0(h_\gamma(\delta)) \subset Sp_0(h_{\gamma^{-1}}(\delta)) .$$

The opposite inclusion is shown in exactly the same way. ■

2.7 Lemma. Let $|\delta| > 1$, $|\delta|\beta^{-1} < |\gamma| < |\delta|\beta$. Then

$$Sp(h_\gamma(\delta)) = Sp_0(h_\gamma(\delta)) ,$$

where $Sp(h_\gamma(\delta))$ denotes the spectrum of $h_\gamma(\delta)$ considered as an operator on the Hilbert space $\ell^2(\mathbb{Z})$.

Proof: First we are going to show that $Sp_0(h_\gamma(\delta)) \subset Sp(h_\gamma(\delta))$. Let $z \in \mathbb{C} \setminus Sp(h_\gamma(\delta))$, and let

$$(h_\gamma(\delta) - z)^{-1} = \sum_{p,q=-\infty}^{\infty} c_{pq}(z) w_{pq}$$

be the “Fourier expansion” of $(h_\gamma(\delta) - z)^{-1}$. Since this expansion decays exponentially in p and q , it defines a bounded linear operator on $c_0(\mathbb{Z})$, which is an inverse of $h_\gamma(\delta) - z$ on $c_0(\mathbb{Z})$. Hence $z \in \mathbb{C} \setminus Sp_0(h(\delta))$. So, $Sp_0(h_\gamma(\delta)) \subset Sp(h_\gamma(\delta))$.

To establish the opposite inclusion, let $z \in \mathbb{C} \setminus Sp_0(h_\gamma(\delta))$. Then $h_\gamma(\delta) - z$ has an inverse S on $c_0(\mathbb{Z})$. Since $Sp_0(h_\gamma(\delta)) = Sp_0(h_{\gamma^{-1}}(\delta))$ by Lemma 2.6, $h_{\gamma^{-1}}(\delta) - z$ also has an inverse on $c_0(\mathbb{Z})$, which we denote by T . Let $(h_{\gamma^{-1}}(\delta) - z)^t$ and T^t be the transposed operators of $h_{\gamma^{-1}}(\delta) - z$ and T , respectively, on the dual Banach space of $c_0(\mathbb{Z})$, which happens to be $\ell^1(\mathbb{Z})$. Since $(h_{\gamma^{-1}}(\delta) - z)^t$ equals the restriction of $h_\gamma(\delta) - z$ on $\ell^1(\mathbb{Z}) \subset c_0(\mathbb{Z})$ and T^t is an inverse of $(h_{\gamma^{-1}}(\delta) - z)^t$ on $\ell^1(\mathbb{Z})$, T^t must be the restriction of S on $\ell^1(\mathbb{Z})$. Now consider the double transposition $(h_\gamma(\delta) - z)^{tt}$ and S^{tt} of $h_\gamma(\delta) - z$ and S , respectively, on the dual Banach space of $\ell^1(\mathbb{Z})$, which happens to be $\ell^\infty(\mathbb{Z})$. Then $(h_\gamma(\delta) - z)^{tt}$ restricted on $c_0(\mathbb{Z})$ equals $h_\gamma(\delta) - z$, while S^{tt} restricted on $c_0(\mathbb{Z})$ equals S . Thus, dispensing with the double t 's, we can summarize the situation as follows:

Considering the linearly embedded Banach spaces $\ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$, we have two linear operators on $\ell^\infty(\mathbb{Z})$, namely, $h_\gamma(\delta) - z$ and S , which are inverses of each other. Both map $\ell^1(\mathbb{Z})$ into $\ell^1(\mathbb{Z})$, and both are continuous on $\ell^\infty(\mathbb{Z})$ as well as on $\ell^1(\mathbb{Z})$ with regard to the respective Banach space norms. In addition, $h_\gamma(\delta) - z$ maps $\ell^2(\mathbb{Z})$ (continuously) into $\ell^2(\mathbb{Z})$. On account of the Riesz-Thorin theorem ([Kn], Ch. IV), we conclude that S maps $\ell^2(\mathbb{Z})$ into $\ell^2(\mathbb{Z})$ and that S restricted on $\ell^2(\mathbb{Z})$ is continuous with respect to the Hilbert space norm. Moreover, S and $h_\gamma(\delta) - z$, restricted on $\ell^2(\mathbb{Z})$, are inverses of each other. Hence $z \in \mathbb{C} \setminus Sp(h(\delta))$, and therefore $Sp(h_\gamma(\delta)) \subset Sp_0(h_\gamma(\delta))$. ■

2.8 Lemma. Let $|\delta| > 1$. Then $Sp(h_\gamma(\delta))$ is constant for $|\delta|^{-1}\beta^{-1} < |\gamma| < |\delta|\beta$.

Proof: The Lemmas 2.3, 2.4 and 2.7 show that $Sp(h_\gamma(\delta))$ is constant for $|\delta|^{-1}\beta^{-1} < |\gamma| < |\delta|\beta$, with the possible exception when $|\gamma| = |\delta|$ and $|\gamma| = |\delta|^{-1}$. Let $Sp(\delta)$ denote the constant spectrum covered by those cases. The lemmas listed also show that

$$Sp(\delta) \subset Sp(h_\gamma(\delta)) \quad \text{for} \quad |\delta|^{-1}\beta^{-1} < |\gamma| < |\delta|\beta.$$

Now consider the “Fourier expansion” of the resolvent of $h(\delta)$

$$(h(\delta) - z)^{-1} = \sum_{p,q=-\infty}^{\infty} c_{pq}(z) w_{pq}.$$

We have $c_{pq}(z) = c_{|p|,q}(z)$. From [R2], paragraph 4, we know that

$$(h_\gamma(\delta) - z)^{-1} = \sum_{p,q=-\infty}^{\infty} \gamma^p c_{pq}(z) w_{pq} \quad \text{for} \quad z \in \mathbb{C} \setminus Sp(h_\gamma(\delta)).$$

These two facts taken together show that $Sp(h_\gamma(\delta))$ increases as $|\log |\gamma||$ increases. In conjunction with the inclusion established above, this shows that $Sp(h_\gamma(\delta)) = Sp(\delta)$ for $|\delta|^{-1}\beta^{-1} < |\gamma| < |\delta|\beta$, as claimed. ■

Remark. The proof of Lemma 2.8 looks a bit twisted. Since it is not this author's aspiration to state and prove the relevant facts in their utmost generality, he feels that the line of reasoning chosen in the context of the objective in this exposition is somewhat adequate. However, it should be mentioned that the claim of Lemma 2.8 is valid for arbitrary irrational numbers α . Here is the brief sketch of a proof.

Consider the set $\mathcal{A}^{(\omega)}$ of analytic elements in \mathcal{A} . An element is called analytic if it has an exponentially decaying "Fourier expansion". $\mathcal{A}^{(\omega)}$ is seen to be a $*$ -subalgebra of \mathcal{A} . Even though the concept of the operator k_γ , as defined at the beginning of this paragraph, cannot be extended to the general setup, the concept of the algebra automorphism $Ad(k_\gamma)$ can. To this end, one needs to show that

$$Sp((e^{\pi\alpha i}\gamma u + \beta)^{-1}(e^{-\pi\alpha i}\gamma^{-1}u + \beta)) = \mathbb{T}, \text{ whenever } \beta^{-1} < |\gamma| < \beta,$$

for all irrational numbers α . This can be fairly easily proved by exploiting the spectral radius formula as well as the ergodicity of the irrational rotation on \mathbb{T} associated with α , in conjunction with Ascoli's theorem. One can use this information to show that the assignments

$$\begin{aligned} u &\longmapsto u \\ v &\longmapsto v(e^{\pi\alpha i}\gamma + \beta)^{-1}(e^{-\pi\alpha i}\gamma^{-1}u^* + \beta) \end{aligned}$$

yield an algebra automorphism $\rho_{(\gamma)}^{(0)}$ of $\mathcal{A}^{(\omega)}$ (which does of course not preserve the involution $*$ unless $|\gamma| = 1$).

Multiplying $\rho_{(\gamma)}^{(0)}$ from both sides by the automorphism $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ yields an automorphism $\rho_{(\gamma)}$ of $\mathcal{A}^{(\omega)}$ which has the property

$$\rho_{(\gamma)}(h_\gamma(\delta)) = h_{\gamma\delta^{-1}}(\delta) \text{ for } \beta^{-1} < |\gamma| < \beta.$$

This identity extends (2.2). Since an element in $\mathcal{A}^{(\omega)}$ is invertible in $\mathcal{A}^{(\omega)}$ if and only if it is invertible in \mathcal{A} , $\rho_{(\gamma)}$ preserves the spectrum of an element in $\mathcal{A}^{(\omega)}$.

Now consider the resolvent $(h_\gamma(\delta) - z)^{-1}$ of $h_\gamma(\delta)$. There is a power series

$$\sum_{p,q=0}^{\infty} c_{pq}(z)x^p y^q$$

in two variables such that

$$(h_\gamma(\delta) - z)^{-1} = \sum_{p,q=-\infty}^{\infty} \tilde{c}_{pq}(z) \gamma^p \delta^q w_{pq} ,$$

where throughout in each of the four (p, q) -quadrants either

$$\tilde{c}_{pq}(z) = c_{|p|,|q|}(z) \quad \text{or} \quad \tilde{c}_{pq}(z) = 0$$

(cf. [R2], paragraph 4). This power series depends on z only and the type of the component in the resolvent set of $h_\gamma(\delta)$ to which z belongs (i.e. bounded component vs. unbounded component). Since $\rho_{(\gamma)}$ preserves the spectrum of any element in $\mathcal{A}^{(\omega)}$, one can show that the domain of convergence of the said power series, which is known to be a logarithmically convex complete Reinhardt domain, must contain the polydisc

$$\{x \in \mathbb{C} \mid |x| \leq d \cdot \beta\} \times \{y \in \mathbb{C} \mid |y| \leq d\} ,$$

where $d = \max\{|\delta|, |\delta|^{-1}\}$. This is valid whenever $|\delta| > 0$ and $d^{-1}\beta^{-1} < |\gamma| < d\beta$. It follows immediately that $Sp(h_\gamma(\delta))$ is constant for $d^{-1}\beta^{-1} < |\gamma| < d\beta$. (Notice that no exception needs to be made for the case $|\delta| = 1$.) ■

We are now in a position to record a preliminary conclusion to our discussion regarding the existence of eigenvectors.

2.9 Proposition. For every $\delta \in \mathbb{C} \setminus (\mathbb{T} \cup \{0\})$ and every $z \in Sp(h(\delta))$ there exist $x \in \mathbb{T}$, $\xi \in \ell^2(\mathbb{Z}) \setminus \{0\}$ such that

$$h(x\delta)\xi = z\xi , \quad \lim_{|n| \rightarrow \infty} |\xi_n|^{1/|n|} \leq \min \{|\delta|\beta^{-1}, |\delta|^{-1}\beta^{-1}\} .$$

Proof: If $|\delta| > 1$, the claim follows from Lemmas 2.3, 2.4, 2.7 and 2.8. Since

$$\mathcal{J}h(\delta)\mathcal{J} = h(\delta^{-1}) , \quad \text{where } (\mathcal{J}\xi)_n = \xi_{-n} ,$$

the case $0 < |\delta| < 1$ can be reduced to the case $|\delta| > 1$. ■

The translation by x in Proposition 2.9, and hence ξ , is (essentially) uniquely determined by z .

2.10 Proposition. If $\delta > 1$; $x, y \in \mathbb{T}$ and

$$h(x\delta)\xi = z\xi, h(y\delta)\eta = z\eta; \xi, \eta \in \ell^2(\mathbb{Z}),$$

then $y = \lambda^{2\ell}x$ for some $\ell \in \mathbb{Z}$ and $u^\ell\xi, \eta$ are linearly dependent.

Proof: We have

$$h_x(x\delta)D_x\xi = zD_x\xi, \quad h_{\overline{y}}(y\delta)D_{\overline{y}}\eta = zD_{\overline{y}}\eta.$$

So, if we let

$$\varphi(a) = \langle aD_x\xi, D_y\overline{\eta} \rangle, \quad a \in \mathcal{A},$$

then

$$\varphi(ah_x(x\delta)) = \varphi(h_y(y\delta)a) = z\varphi(a).$$

Now let

$$\varphi_{pq} = s^{p+q}\delta^{-q}\varphi(w_{pq}), \quad \text{where } s^2 = \overline{xy}.$$

Then the double sequence $\{\varphi_{pq}\}$ is seen to solve the system of difference equations

$$\begin{aligned} (*) \quad & \cos(\pi\alpha q + \theta)(X_{p-1,q} + X_{p+1,q}) + \beta \cos(\pi\alpha p + \theta)(X_{p,q-1} + X_{p,q+1}) = zX_{pq} \\ & \sin(\pi\alpha q + \theta)(X_{p-1,q} - X_{p+1,q}) - \beta \sin(\pi\alpha p + \theta)(X_{p,q-1} - X_{p,q+1}) = 0, \end{aligned}$$

where $e^{i\theta} = s$. Starting over again with $\overline{\xi}$ and $\overline{\eta}$ in place of ξ and η , respectively, we get

$$h_x(x\delta^{-1})D_x\overline{\xi} = \overline{z}D_x\overline{\xi}, \quad h_{\overline{y}}(y\delta^{-1})D_{\overline{y}}\overline{\eta} = \overline{z}D_{\overline{y}}\overline{\eta}.$$

Let

$$\psi(a) = \langle aD_x\overline{\xi}, D_y\eta \rangle, \quad a \in \mathcal{A}.$$

Now let

$$\psi_{pq} = s^{p+q}\delta^q\psi(w_{pq}), \quad s \text{ as before.}$$

Replacing z by \overline{z} in $(*)$ it is seen that the double sequence $\{\psi_{pq}\}$ solves $(*)$. So both, $\{\varphi_{pq}\}$ and $\{\overline{\psi}_{pq}\}$ solve $(*)$ for the same parameter z . Moreover, the sequence

$$\{\varphi_{pp}\psi_{p+1,p+1} - \varphi_{p+1,p+1}\psi_{pp}\}$$

is bounded.

Now suppose that the claim of the proposition is not true. Then it follows from appendix A1 that $\{\varphi_{pq}\}$ and $\{\overline{\psi}_{pq}\}$ are linearly dependent. This however implies that $\{\varphi_{pq}\}$ decays exponentially uniformly in p of order at least δ^{-2} as $q \rightarrow \infty$. Since the sequence $\{\psi_{pq}\}_{q \in \mathbb{Z}}$ is almost periodic for every $p \in \mathbb{Z}$, and hence does not approach zero as $q \rightarrow \infty$, we have reached a contradiction. \blacksquare

Our next objective is to show that the spectrum of $h = u^* + u + \beta(v + v^*)$ is a regular compactum in the sense of potential theory. In view of [R3], Theorem 2.2 we are going to prove a stronger statement. In preparation of this, we need the following.

2.11 Lemma. $Sp(h) \subset \mathbb{C} \setminus Sp(h(\delta))$ whenever $|\delta| \neq 1$.

Proof: Suppose this were not true. Then there exists $\chi \in Sp(h)$ and $\delta_0 > 1$ such that $\chi \in Sp(h(\delta))$ for $\delta_0^{-1} \leq \delta \leq \delta_0$. According to Proposition 2.9, for any $\delta \in [\delta_0^{-1}, \delta_0] \setminus \{1\}$, there exist $x \in \mathbb{T}$ and $\xi \in \ell^2(\mathbb{Z}) \setminus \{0\}$ such that

$$h(x\delta)\xi = \chi\xi, \quad \overline{\lim}_{|n| \rightarrow \infty} |\xi|^{1/|n|} \leq \delta^{-1}\beta^{-1},$$

For every $y \in \mathbb{T}$, let

$$\eta_n^{(y)} = \sum_{p=-\infty}^{\infty} y^p x^{-n} \delta^{-n} \langle w_{pn}\xi, \mathcal{J}\xi \rangle w_{pn} e^{(0)},$$

where (\mathcal{J} as in 2.9),

$$e_n^{(0)} = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

Let $\widehat{h}(y) = u + u^* + \beta^{-1}(yv + \overline{y}v^*)$. Then

$$\widehat{h}(y)\eta^{(y)} = \chi\eta^{(y)}, \quad \overline{\lim}_{n \rightarrow \infty} |\eta_n^{(y)}|^{1/n} \leq \delta^{-1}, \quad \overline{\lim}_{n \rightarrow -\infty} |\eta_n^{(y)}|^{1/|n|} \leq \delta.$$

Exempting for every $\delta \in [\delta_0^{-1}, \delta_0] \setminus \{1\}$ a possibly non-empty, but countable subset of values for the parameter y , for which $\eta^{(y)}$ might be zero, this implies the following: There is a countable subset $M \subset \mathbb{T}$ and a dense countable subset $N \subset [\delta_0^{-1}, \delta_0] \setminus \{1\}$ such that for every $y \in \mathbb{T} \setminus M$ and every $\delta \in N$ there exists $\eta \in \mathbb{C}^\infty \setminus \{0\}$ such that

$$\widehat{h}(y)\eta = \chi\eta, \quad \overline{\lim}_{n \rightarrow \infty} |\eta_n|^{1/n} \leq \delta^{-1}, \quad \overline{\lim}_{n \rightarrow -\infty} |\eta_n|^{1/|n|} \leq \delta.$$

This in turn entails that for every $y \in \mathbb{T} \setminus M$, the operator $\widehat{h}(y)$ has an exponentially decaying eigenvector for χ . Since $\widehat{h}(y)$ is known to have no eigenvalues for any $y \in \mathbb{T}$, we have reached a contradiction. \blacksquare

Letting μ be the probability measure of $Sp(h)$ obtained through restricting the canonical trace τ of \mathcal{A} on the C^* -algebra generated by h , Lemma 2.11 entails, on account of [R3], Theorem 2.2:

2.12 Proposition. $Sp(h(\delta)) = \{z \in \mathbb{C} \mid \int \log |z - s| d\mu(s) = \log(\beta\delta)\}$ for $\delta \geq 1$. In particular $Sp(h)$ is a regular compactum and μ is its equilibrium distribution.

2.13 Corollary. $Sp(h)$ is not connected.

Proof: Consider the moments $\tau(h^n)$ of h . While $\tau(h^{2n-1}) = 0$ for every $n \in \mathbb{N}$, elementary calculations show that

$$\tau(h^2) = 2\beta^2 + 2, \quad \tau(h^4) = 6\beta^4 + (24 + 16 \cos 2\pi\alpha)\beta^2 + 6.$$

Now consider $v + v^* = \lim_{\beta \rightarrow \infty} \beta^{-1}(u + u^* + \beta(v + v^*))$. This element has a connected spectrum and τ restricted to the C^* -algebra generated by $v + v^*$ is nothing but the equilibrium distribution for $Sp(v + v^*) = [-2, 2]$. Again we have $\tau(v + v^*)^{2n-1} = 0$ for every $n \in \mathbb{N}$. Moreover,

$$\tau((v + v^*)^2) = 2, \quad \tau((v + v^*)^4) = 6.$$

In order to identify possible values for β for which $Sp(h)$ is connected, we have to solve the equations

$$2x = \tau(h^2), \quad 6x^2 = \tau(h^4),$$

for x and β . Eliminating x , we obtain

$$3\tau(h^2)^2 = 2\tau(h^4),$$

which in turn yields

$$(1 + \cos 2\pi\alpha)\beta^2 = 0.$$

Since α is irrational this is valid only for $\beta = 0$. Note that $\beta = 0$ does indeed correspond to an element with a connected spectrum, namely $u + u^*$, which is the image of $v + v^*$ under a “Fourier transform”. ■

We will show now that the eigenvectors of the operators $h(\delta)$ depend continuously on the spectral parameter z in a sense to be made precise below.

2.14 Proposition. Let $d > 1$, and for every $m \in \mathbb{N}$ let $|\delta_m| \geq d$, $z_m \in Sp(h(\delta_m))$, $\xi^{(m)} \in \ell^2(\mathbb{Z}) \setminus \{0\}$ such that

$$h(\delta_m)\xi^{(m)} = z_m\xi^{(m)}, \quad \overline{\lim}_{|n| \rightarrow \infty} \left| \xi_n^{(m)} \right|^{1/|n|} \leq |\delta_m|^{-1}\beta^{-1}$$

and $\lim_{m \rightarrow \infty} z_m = z$.

Then there exist $\delta \in \mathbb{C}$, $\xi \in \ell^2(\mathbb{Z}) \setminus \{0\}$ and $j_m \in \mathbb{N}$, $c_m \in \mathbb{C} \setminus \{0\}$ such that $h(\delta)\xi = z\xi$, $\lim_{m \rightarrow \infty} \lambda^{2j_m} \delta_m = \delta$ and

$$\lim_{m \rightarrow \infty} \left\| D_\gamma \left(c_m u^{j_m} \xi^{(m)} - \xi \right) \right\|_\infty = 0 \quad \text{uniformly for } d^{-1} \leq \gamma \leq d.$$

Proof: It follows from (2.2) that there exists $t_m \in \mathbb{C}$ such that

$$k D_{\delta_m} \xi^{(m)} = t_m D_{\delta_m}^{-1} \xi^{(m)}.$$

Let

$$\eta^{(m)} = \|D_{\delta_m} \xi^{(m)}\|_\infty^{-1} D_{\delta_m} \xi^{(m)}.$$

Then $\|\eta^{(m)}\|_\infty = 1$ and there exist $j_m \in \mathbb{N}$ such that

$$\left| \eta_{j_m}^{(m)} \right| \geq \frac{1}{2}.$$

Adjusting $\eta^{(m)}$ suitably through shifting and multiplication by scalars, we may assume that

$$\eta_0^{(m)} \geq \frac{1}{2}.$$

Finally, switching to a subsequence if necessary, we may assume that the sequence $\eta^{(1)}, \eta^{(2)}, \dots$ converges weakly to some $\eta \in \ell^\infty(\mathbb{Z}) \setminus \{0\}$. Adjusting the t_m accordingly we have

$$k \eta^{(m)} = t_m D_{\delta_m}^{-2} \eta^{(m)}.$$

Since $k\eta \neq 0$, it follows that $\inf\{|t_m| \mid m \in \mathbb{N}\} > 0$. Also, since Proposition 2.12 implies that $|\delta_1|, |\delta_2|, \dots$ is convergent, $\sup\{|t_m| \mid m \in \mathbb{N}\} < \infty$. For, if this were not the case, then we could find some point of density δ for the δ_m such that $D_\delta^{-2} \eta = 0$.

Switching once again to a subsequence if necessary, we may assume that the sequences $\{\delta_m\}$ and $\{t_m\}$ are convergent. Let $\delta = \lim_{m \rightarrow \infty} \delta_m$. Since k defines a bounded operator on $\ell^\infty(\mathbb{Z})$, it follows that the corresponding sequence of the $\xi^{(m)}$, after having been suitably scaled and adjusted through shifts, does indeed converge uniformly to some eigenvector ξ of $h(\delta)$ for z , as claimed in the proposition. Since eigenvectors of $h(\delta)$ are essentially unique by Proposition 2.10, the conclusion of the proposition applies to the original sequence of the $\xi^{(m)}$ as well. ■

Remark: Tracing the steps in the proof of Lemmas 2.1 and 2.2 one can improve the convergence of eigenvectors in Proposition 2.14 to the effect that γ may range over any closed interval contained in $(d^{-1}\beta^{-1}, d\beta)$. ■

Our next objective is to refine Propositions 2.9 and 2.13 by showing that there is a natural parametrization of the eigenvalue problem for the operators $h(\delta)$, $|\delta| > 1$, through a Riemann surface $\tilde{\mathcal{R}}$ covering the resolvent set $\mathcal{R} = \mathbb{C} \setminus Sp(h)$. By doing so we will discover that all the eigenvectors of these operators correspond to orbits of a cyclic group of covering transformations on $\tilde{\mathcal{R}}$, while the square of their components can be obtained through evaluation of a single analytic function on $\tilde{\mathcal{R}}$. For the basic concepts of Riemann surfaces we are going to use and for the terminology that comes with it, we refer to [AS], Chapters I and II.

Let I be the smallest interval containing $Sp(h)$. There exists a unique analytic function G on $\mathbb{C} \setminus I$ such that

$$\log |G(z)| = \int \log |z - s| d\mu(s) - \log \beta, \quad G(\mathbb{R}^+ \setminus Sp(h)) \subset \mathbb{R}^+.$$

If f is a closed arc surrounding a component K of $Sp(h)$, then any two analytic continuations of G along f over the same point z on f differ by a multiplicative constant of the form

$$e^{2\pi\mu(K)ni}.$$

A well-known fact in the K -theory of the irrational rotation C^* -algebra provides us with the information that

$$\mu(K) \in (\mathbb{Z} + \alpha\mathbb{Z}) \cap [0, 1],$$

so that the said multiplicative factor takes the form λ^{2n} . Now G can be continued analytically along any arc in $\mathcal{R} = \mathbb{C} \setminus Sp(h)$. So, if we let \mathcal{F} be the universal covering of \mathcal{R} , \mathcal{G} the corresponding group of covering transformations, and finally \tilde{G} the analytic function on \mathcal{F} obtained through analytic continuation of G , then we have for every $g \in \mathcal{G}$

$$\tilde{G} \circ g = \lambda^{2n} \tilde{G} \text{ for some } n \in \mathbb{Z}.$$

Let

$$\mathcal{G}_0 = \{g \in \mathcal{G} \mid \tilde{G} \circ g = \tilde{G}\}.$$

Then \mathcal{G}_0 is a normal subgroup of \mathcal{G} with a cyclic quotient group $\mathcal{G}/\mathcal{G}_0$. The \mathcal{G}_0 -orbits in \mathcal{F} form a Riemann surface $\tilde{\mathcal{R}}$ which covers \mathcal{R} , and whose group of covering transformations corresponds to $\mathcal{G}/\mathcal{G}_0$ in a natural way. Let p be the covering map of $\tilde{\mathcal{R}}$ over \mathcal{R} . Finally,

\tilde{G} defines an analytic map on $\tilde{\mathcal{R}}$ which we denote by G again. We assume ω to be that generator of the group of covering transformations of $\tilde{\mathcal{R}}$ over \mathcal{R} for which $G(\omega(z)) = \lambda^2 G(z)$.

We are now in a position to state the major claim in this paragraph. For its proof, we need another technical lemma.

2.15 Lemma. Suppose $f : \tilde{\mathcal{R}} \rightarrow \mathbb{T}$ has the following properties

- (i) Every subsequence of $\{f^n \mid n \in \mathbb{Z}\}$ has a subsequence which converges uniformly on compact subsets of $\tilde{\mathcal{R}}$.
- (ii) If $\lim_{n \rightarrow \infty} z_n = z$ and $\lim_{n \rightarrow \infty} f(z_n) = c$, then $c \in \{f(z), \lambda^2 f(z)\}$.

Then $f(\tilde{\mathcal{R}}) \subset \{x\lambda^{2n} \mid n \in \mathbb{Z}\}$ for some $x \in \mathbb{T}$.

Proof: Let $K \subset \tilde{\mathcal{R}}$ be compact and connected. It suffices to show that $f(K) \subset \{x\lambda^{2n} \mid n \in \mathbb{Z}\}$ for some $x \in \mathbb{T}$. Let Ω be the uniform closure in $\ell^\infty(K)$ of $\{g^n \mid n \in \mathbb{Z}\}$, where $g = f/k$. It follows from (i) that Ω is a metrizable and compact group. To every point y in $g(K)$ there corresponds a character φ_y of Ω , a continuous homomorphism from Ω into \mathbb{T} , such that $\varphi_y(g) = y$. Let $\mathcal{Y} = \{\varphi_y \mid y \in g(K)\}$. Since the uniform structure on Ω is inherited from $\ell^\infty(K)$, the set \mathcal{Y} is equicontinuous on Ω . Hence $\overline{\mathcal{Y}}$, the closure of \mathcal{Y} in the topology of uniform convergence on Ω , is compact. In other words, $\overline{\mathcal{Y}}$ is a compact subset of the dual group $\hat{\Omega}$ of Ω . Since Ω is compact, $\hat{\Omega}$ is discrete. Therefore $\overline{\mathcal{Y}}$, being a compact subset of $\hat{\Omega}$, must be finite. It follows that $g(K)$ is a finite subset of \mathbb{T} .

For $x \in \mathbb{T}$, let $\mathcal{O}_x = \{x\lambda^{2n} \mid n \in \mathbb{Z}\}$. Now let $x \in \mathbb{T}$ be such that

$$M = g(K) \cap \mathcal{O}_x \neq \emptyset.$$

Let $N = g(K) \setminus \mathcal{O}_x$. Since $g(K)$ is finite, M and N are finite as well and hence, closed. Therefore, since K is compact, property (ii) implies

$$\overline{g^{-1}(M)} \subset M \cup \lambda^2 M, \quad \overline{g^{-1}(N)} \subset N \cup \lambda^2 N.$$

By the definition of M and N

$$(M \cup \lambda^2 M) \cap (N \cup \lambda^2 N) = \emptyset.$$

Hence,

$$\overline{g^{-1}(M)} \cap \overline{g^{-1}(N)} = \emptyset, \quad \overline{g^{-1}(M)} \cup \overline{g^{-1}(N)} = K.$$

Since K is connected by assumption,

$$g^{-1}(M) = \overline{g^{-1}(M)} = K ,$$

which settles the claim. ■

2.16 Theorem. For every $z \in \tilde{\mathcal{R}}$, there exists $\xi^{(z)} \in \ell^2(\mathbb{Z}) \setminus \{0\}$ such that

$$h(G(z))\xi^{(z)} = p(z)\xi^{(z)} , \quad \overline{\lim_{|n| \rightarrow \infty}} \left| \xi_n^{(z)} \right|^{1/|n|} \leq |G(z)|^{-1} \beta^{-1} ,$$

and for every $p \in \mathbb{Z}$, $\vartheta_p(z) = \xi_0^{(z)} \xi_p^{(z)}$ is an analytic function on $\tilde{\mathcal{R}}$ with the property

$$\vartheta_p(\omega^n(z)) = \xi_n^{(z)} \xi_{n+p}^{(z)} .$$

Proof: Propositions 2.9 and 2.12 taken together imply that for every $z \in \tilde{\mathcal{R}}$, there exist $x_z \in \mathbb{T}$, $\xi^{(z)} \in \ell^2(\mathbb{Z}) \setminus \{0\}$ such that

$$h(x_z G(z))\xi^{(z)} = p(z)\xi^{(z)} , \quad \overline{\lim_{|n| \rightarrow \infty}} \left| \xi_n^{(z)} \right|^{1/|n|} \leq |G(z)|^{-1} \beta^{-1} .$$

On account of (2.2), there exists for every $z \in \tilde{\mathcal{R}}$ a $t_z \in \mathbb{C}$ such that

$$(*) \quad kD_{x_z G(z)} \xi^{(z)} = t_z D_{x_z G(z)}^{-1} \xi^{(z)} .$$

Applying u^m to both sides of this identity yields

$$kD_{x_z G(z)} u^m \xi^{(z)} = (x_z G(z))^2 t_z D_{x_z G(z)}^{-1} u^m \xi^{(z)} .$$

Therefore, we can adjust x_z , $\xi^{(z)}$ and t_z in $(*)$ such that

$$1 \leq |t_z| < |G(z)|^2 .$$

It follows from Proposition 2.10 that

$$\lim_{n \rightarrow \infty} z_n = z , \quad \lim_{n \rightarrow \infty} |t_{z_n}| = s \quad \text{and} \quad \lim_{n \rightarrow \infty} x_{z_n} = y$$

implies

$$s \in \{|t_z|, |G(z)|^2 |t_z|\} , \quad y \in \{x_z, \lambda^2 x_z\} .$$

Now let

$$\varphi^{(z)}(a) = \langle a\xi^{(z)}, \overline{\xi^{(z)}} \rangle, \quad a \in \mathcal{A}.$$

Then

$$\varphi^{(z)}(h(x_z G(z))a) = \varphi(ah(x_z G(z))) = p(z)\varphi^{(z)}.$$

So, if we define

$$\psi_{pq}^{(z)} = \overline{x}_z^q G(z)^{-q} \varphi(w_{pq}); \quad p, q \in \mathbb{Z},$$

then the double sequence $\{\psi_{pq}^{(z)}\}$ is a solution to the system of difference equations in the Appendix A2. Also, $\{\psi_{pq}^{(z)}\}$ decays exponentially uniformly in p with an order less than or equal to β^{-1} . Therefore, it follows from A2 that, after scaling the eigenvectors $\xi^{(z)}$ suitably,

$$(**) \quad \lambda^{pq} \overline{x}_z^q \sum_{n=-\infty}^{\infty} \overline{\lambda}^{2qn} \xi_n^{(z)} \xi_{n+p}^{(z)} = (c_{pq}(p(z)) - d_{pq}(p(z)) G(z)^q); \quad p, q \in \mathbb{Z},$$

where c_{pq} and d_{pq} are analytic functions on \mathcal{R} which are determined by the identities

$$\begin{aligned} h - x)^{-1} &= \sum_{p,q=-\infty}^{\infty} c_{pq}(x) w_{pq} \quad \text{and} \\ (h(\delta) - x)^{-1} &= \sum_{p,q=-\infty}^{\infty} d_{pq}(x) \delta^q w_{pq}, \quad |\delta| > |G(z)|. \end{aligned}$$

If $d > 1$ and $K \subset \{z \in \widetilde{\mathcal{R}} \mid |G(z)| \geq d\}$ is compact, then Proposition 2.11 ensures that

$$\|D_\gamma \xi^{(z)}\|_\infty \text{ is uniformly bounded in } z \in K, d^{-1} \leq |\gamma| \leq d.$$

Since the functions on the right hand side of (2.4) are analytic on \mathcal{R} as well as uniformly bounded on K in p and q , we conclude that if we let $f(z) = \overline{x}_z$, then any subsequence of $\{f^q\}$ has a subsequence which is locally uniformly convergent. Thus we have show that f has the properties (i) and (ii) in Lemma 2.15. It follows that

$$f(\widetilde{\mathcal{R}}) \subset \{\overline{y} \lambda^{2n} \mid n \in \mathbb{Z}\} \text{ for some } y \in \mathbb{T}.$$

By applying suitable powers of u to the vectors $\xi^{(z)}$, we can adjust $(**)$ to the effect that $x_z = y$ for every $z \in \widetilde{\mathcal{R}}$.

Let $t > \max Sp(h)$. Then there exists $z_t \in \widetilde{\mathcal{R}}$ such that $G(z) \in \mathbb{R}^+$. Hence

$$h(yG(z_t))\xi^{(z_t)} = t\xi^{(z_t)}, \quad h(\overline{y}G(z_t))\overline{\mathcal{J}\xi^{(z_t)}} = t\overline{\mathcal{J}\xi^{(z_t)}}$$

and by Proposition 2.10, $\bar{y} = \lambda^{2m}y$ for some $m \in \mathbb{Z}$. Thus, adjusting the eigenvectors through a suitable power of the shift we may assume that $y \in \{1, -1, \lambda, -\lambda\}$. In order to show that y equals actually 1, one can proceed as follows: Since $\beta^{-1}G(z_t)h(yG(z_t))$ approaches yv as $t \rightarrow \infty$, one can use (**) to show that $\xi^{(z_t)}$ approaches an eigenvector η of yv (in $\ell^2(\mathbb{Z})$, say) for some positive eigenvalue s . Thus $s \in \{\bar{y}\lambda^{2n} \mid n \in \mathbb{Z}\}$. The only way this can happen is when $y = 1$.

Proposition 2.13 in conjunction with (**) entails that the function

$$\psi_p : \mathbb{T} \times \tilde{\mathcal{R}} \rightarrow \mathbb{C} ; \quad \psi_p(x, z) = \sum_{n=-\infty}^{\infty} x^n \xi_n^{(z)} \xi_{n+p}^{(z)}$$

is continuous and analytic in z for every $x \in \mathbb{T}$. It follows that the functions

$$\vartheta_p(z) = \int \psi_p(x, z) dx = \xi_0^{(z)} \xi_p^{(z)}$$

are analytic. Moreover, since $G(\omega(z)) = \lambda^2 G(z)$,

$$\vartheta(\omega^n(z)) = \xi_n^{(z)} \xi_{n+p}^{(z)}$$

by (**), as claimed. ■

Remarks. 1) The second part of Theorem 2.16 says (take $p = 0$), that given $\xi_0^{(z)}$ in a neighborhood of z , one can generate $\left(\xi_n^{(z)}\right)^2$ through analytic continuation of $\left(\xi_0^{(z)}\right)^2$.

2) Theorem 2.16 also shows that we can identify the points of the Riemann surface $\tilde{\mathcal{R}}$ with the one-dimensional eigenspaces of the operators $h(G(z))$ for $p(z)$ in such a way that the covering transformation ω corresponds to the two-sided shift on $\ell^2(\mathbb{Z})$.

3) From (**) in the proof of Theorem 2.16 we can extract the identity

$$\sum_{n=-\infty}^{\infty} \left(\xi_n^{(z)}\right)^2 = \int (t - p(z))^{-1} d\mu(t) .$$

It follows that $\sum_{n=-\infty}^{\infty} \left(\xi_n^{(z)}\right)^2$ equals zero if and only if z is a critical point for the conductor potential of $Sp(h)$. These critical points are simple; and exactly one is located in every gap of $Sp(h)$ and nowhere else. This shows that the eigenvectors $\xi^{(z)}$ carry the relevant information regarding the gap structure for $Sp(h)$ in a very explicit form.

4) Elaborating on the comments made above, there are some implications for the case $|\delta| = 1$. One might be tempted to try to generate exponentially decaying eigenvectors for the operators $h(x)$, $|x| = 1$, as (uniform) limits of the eigenvectors $\xi^{(z)}$. Assuming that $Sp(h)$ has infinitely many gaps, one is confronted with the following impediment: Let z_1, z_2, \dots be a sequence of critical points converging to $\chi \in Sp(h)$. Suppose that $h(x)\xi = \chi\xi$ for some $\xi \in \ell^2(\mathbb{Z}) \setminus \{0\}$. Then

$$\sum_{n=-\infty}^{\infty} \xi_n^2 = y \|\xi\|_2^2 \neq 0 \quad \text{for some } y \in \mathbb{T}.$$

It follows that ξ cannot be approximated by the (suitably scaled) sequence $\xi^{(z_1)}, \xi^{(z_2)}, \dots$ in the Hilbert space norm.

2.17 Corollary. There exist homeomorphisms σ and ι on $\tilde{\mathcal{R}}$ such that

$$\begin{aligned} \xi^{(\sigma(z))} &= \mathcal{J} \overline{\xi^{(z)}} \quad , \quad p(\sigma(z)) = \overline{p(z)} \quad , \quad G(\sigma(z)) = \overline{G(z)} \\ \xi^{(\iota(z))} &= D_{-1} \xi^{(z)} \quad , \quad p(\iota(z)) = -p(z) \quad , \quad G(\iota(z)) = -G(z) \end{aligned}.$$

In particular, $\sigma^2 = \iota^2 = Id$, $\sigma \circ \omega = \omega^{-1} \circ \sigma$, $\omega \circ \iota = \iota \circ \omega$.

Proof: This follows immediately from Theorem 2.16, noting that

$$h(\overline{G(z)}) \mathcal{J} \overline{\xi^{(z)}} = \overline{p(z)} \mathcal{J} \overline{\xi^{(z)}}$$

and

$$h(-G(z)) D_{-1} \xi = -\overline{p(z)} D_{-1} \xi^{(z)} . \quad \blacksquare$$

The following is in preparation for the discussion in paragraph 3.

2.18 Corollary. There exists an analytic function Γ on $\tilde{\mathcal{R}}$ which has the following properties:

$$\begin{aligned} k D_{G(z)} \xi^{(z)} &= \Gamma(z) D_{G(z)-1} \xi^{(z)} \quad , \quad \Gamma(\omega(z)) = G(z)^2 \Gamma(z) \quad , \quad \Gamma(\sigma(z)) = \overline{\Gamma(z)}^{-1} \\ \text{and } \Gamma(\iota(z)) &= \Gamma(z). \end{aligned}$$

Proof: There clearly exists a function satisfying the stated identities. To show that it is actually analytic, we consider scalar products

$$\langle k D_{G(z)} \xi^{(z)} , a \overline{\xi^{(z)}} \rangle = \Gamma(z) \langle D_{G(z)-1} \xi^{(z)} , a \overline{\xi^{(z)}} \rangle ,$$

where a is any linear combination of the elements w_{pq} . Written out in components, one can see that the function on the left as well as the second factor on the right are algebraically generated by G , G^{-1} and functions of the form $\vartheta_p \circ \omega^n$, all of which are analytic. Therefore, Γ has to be analytic as well. \blacksquare

3. The kernel of a family of related operators

The first identity in Corollary 2.18 gives rise to an eigenvalue problem in its own right, involving the operator k and the function Γ as an eigenvalue parameter. The question arises whether the two related eigenvalue problems are actually equivalent. It will be shown that this is essentially the case. More specifically, it will be shown that to any two distinct points z_1 and z_2 in the complement of a (possibly empty) discrete subset $\tilde{\mathcal{R}}$, there correspond distinct pairs of parameters $(G(z_1), \Gamma(z_1))$ and $(G(z_2), \Gamma(z_2))$. To this end, we introduce new operators whose kernel will hold all the relevant information.

Let g be the function defined in (1.6) and let $\gamma \in \mathbb{R}$ be close to 1 having the property that

$$g(\gamma\mathbb{T}) \cap \left(\left\{ \pi \left(n + \frac{1}{2} \right) \mid n \in \mathbb{Z} \right\} \cup \left\{ \pi \left(m + i \left(n + \frac{\pi}{4} \right) \right) \mid m, n \in \mathbb{Z} \right\} \right) = \emptyset .$$

Let Γ be as in Corollary 2.18 and let

$$\Omega = \{z \in \tilde{\mathcal{R}} \mid 1 + \Gamma(\omega^n(z)) \neq 0 \text{ for all } n \in \mathbb{Z}\} .$$

Since Γ is analytic, Ω is a discrete subset of $\tilde{\mathcal{R}}$. In the following, Ω will be augmented as needed, but it will always be discrete. Now we define for every $z \in \tilde{\mathcal{R}} \setminus \Omega$ a bounded operator $H_\gamma(z)$ on the Hilbert space $\ell^2(\mathbb{Z})$,

$$(H_\gamma(z)\xi)_n = \sum_{j=-\infty}^{\infty} a_j \xi_{n+j} + i \frac{1 - \Gamma(\omega^n(z))}{1 + \Gamma(\omega^n(z))} \xi_n ,$$

where

$$\tan g(\gamma u) = \sum_{j=-\infty}^{\infty} a_j u^j .$$

The sequence $\{a_j\}$ decays exponentially as $|j| \rightarrow \infty$. Obviously, $H(z)$ is analytic in z .

Moreover,

$$(3.1) \quad \begin{cases} H_\gamma(\omega(z)) = u H_\gamma(z) u^* \\ H_\gamma(\sigma(z)) = \mathcal{J} H_{\gamma^{-1}}(z)^* \mathcal{J} . \end{cases}$$

3.1 Proposition. The operator $H(z)$ is Fredholm with index zero. Any two of these operators are compact perturbations of each other, and their essential spectrum equals

$$(\tan g(\gamma\mathbb{T}) + i) \cup (\tan g(\gamma\mathbb{T}) - i) .$$

Proof: Let $T \in \mathcal{B}(\ell^2(\mathbb{Z}))$ be defined as follows:

$$(T\xi)_n = \sum_{j=-\infty}^{\infty} a_j \xi_{n+j} + \mathcal{E}(n) \xi_n ,$$

where

$$\mathcal{E}(n) = \begin{cases} -i & , \quad n \geq 0 \\ i & , \quad n < 0. \end{cases}$$

Since $|\Gamma(\omega^n(z))| = |G(z)^{2n}\Gamma(z)|$ and $|G(z)| > 1$,

$$\lim_{n \rightarrow \infty} i \frac{1 - \Gamma(\omega^n(z))}{1 + \Gamma(\omega^n(z))} = -i \quad , \quad \lim_{n \rightarrow -\infty} i \frac{1 - \Gamma(\omega^n(z))}{1 + \Gamma(\omega^n(z))} = i \quad .$$

It follows that $H(z)$ is a compact perturbation of T . Let

$$T_0 = T - \mathcal{E} \quad ,$$

and let $\tilde{T}_0, \tilde{\mathcal{E}}$ be the range of T_0, \mathcal{E} , respectively, in the Calkin algebra $\mathcal{B}(\ell^2(\mathbb{Z}))/\mathcal{K}$, where \mathcal{K} denotes the C^* -algebra of compact operators on $\ell^2(\mathbb{Z})$. Then \tilde{T}_0 and $\tilde{\mathcal{E}}$ are normal operators which commute and their joint spectrum equals

$$\tan g(\mathbb{T}) \times \{-i, i\} \quad .$$

It follows that

$$Sp\left(\tilde{T}_0 + \tilde{\mathcal{E}}\right) = (\tan g(\gamma\mathbb{T}) + i) \cup (\tan g(\gamma\mathbb{T}) - i)$$

which equals the essential spectrum of T . By our choice of γ , $\tilde{T}_0 + \tilde{\mathcal{E}}$ is invertible, which entails that $H(z)$ is Fredholm with index zero. ■

In the following we will constantly make use of Theorem 2.16. In particular, the eigenvectors which occur will be assumed to decay of a sufficiently high order, so that all manipulations make sense, unless specified otherwise. The significance of the operators $H(z)$ rests with the following statement.

3.2 Proposition. Suppose $h_\gamma(G(z))\xi = p(z)\xi$ for some $z \in \tilde{\mathcal{R}} \setminus \Omega$. Then there exist $t_n \in \mathbb{C}$ such that

$$t_n^2 = \Gamma(\omega^n(z)) \quad , \quad H_\gamma(z)\eta = 0 \quad ,$$

where

$$\eta_n = (t_n + t_n^{-1}) \xi_n .$$

Proof: By Corollary 2.18

$$k_\gamma D_{G(z)} \xi = \Gamma(z) D_{G(z)^{-1}} \xi .$$

Rearranging the diagonal operators involved, we obtain

$$T e^{ig(\gamma u)} \xi = T^{-1} \xi$$

where T is a diagonal operator such that $(T\xi)_n = t_n \xi_n$ with $t_n^2 = \Gamma(\omega^n(z))$. Using the identity

$$e^{ig(\gamma u)} = \frac{1 + i \tan \frac{g(\gamma u)}{2}}{1 - i \tan \frac{g(\gamma u)}{2}}$$

we have

$$\left(1 + i \tan \frac{g(\gamma u)}{2}\right) T \xi = \left(1 - i \tan \frac{g(\gamma u)}{2}\right) T^{-1} \xi$$

or

$$t_n \xi_n + i \sum_{j=-\infty}^{\infty} a_j t_{n+j} \xi_{n+j} = t_n^{-1} \xi_n - i \sum_{j=-\infty}^{\infty} a_j t_{n+j}^{-1} \xi_{n+j}$$

which turns into

$$\sum_{j=-\infty}^{\infty} a_j \eta_{n+j} + i \frac{t_n^{-1} - t_n}{t_n^{-1} + t_n} \eta_n = 0 .$$

Since $t_n^2 = \Gamma(z)$, the second term equals

$$i \frac{1 - \Gamma(\omega^n(z))}{1 + \Gamma(\omega^n(z))} . \quad \blacksquare$$

Remark: If we replace ξ by $\tilde{\xi} = D_{-1} \xi$ in Proposition 3.2, then

$$h(G(\iota(z))) \tilde{\xi} = -p(z) \tilde{\xi} .$$

But if we transform $\tilde{\xi}$ analogous to ξ , thus obtaining a vector $\tilde{\eta}$, we have $\eta = \tilde{\eta}$. Succinctly put, one can say that to every non-zero element in the kernel of $H_\gamma(z)$ there correspond eigenvectors of $h(G(z))$ and $h(G(\iota(z)))$, respectively, whose eigenvalues differ by a negative sign. Also note that $H(\iota(z)) = H(z)$, since $\Gamma(\iota(z)) = \Gamma(z)$. This means that the operators $H(z)$ are more appropriately parametrized through the Riemann surface obtained from $\tilde{\mathcal{R}}$ by identifying z and $\iota(z)$. ■

Propositions 3.1 and 3.2 taken together say that $H(z)$ is a Fredholm operator of index zero with a non-trivial kernel and that 0 is an isolated point in the spectrum of $H_\gamma(z)$ for every $z \in \tilde{\mathcal{R}} \setminus \Omega$. This situation provides the proper setting for the employment of analytic perturbation theory of linear operators as expounded in [Ko], for instance.

First we enlarge Ω by the set of points $z \in \tilde{\mathcal{R}}$ for which the following is true: For every neighborhood $\mathcal{U} \subset \tilde{\mathcal{R}}$ of z and for every neighborhood $\mathcal{V} \subset \mathbb{C}$ of 0, there exists $\tilde{z} \in \mathcal{U}$ such that $H_\gamma(\tilde{z})$ has a non-zero eigenvalue in \mathcal{V} . Since $H_\gamma(z)$ is analytic in z , the inclusion of those branch-points in Ω still yields a discrete subset of $\tilde{\mathcal{R}}$.

For every point $z \in \tilde{\mathcal{R}} \setminus \Omega$, let

$$P(z) = \frac{1}{2\pi i} \oint (t - H_\gamma(z))^{-1} dt ,$$

where the integral is taken over a positively-oriented circle enclosing 0 but no other point in the spectrum of $H_\gamma(z)$. Then $P(z)$ is a projection of finite rank. Moreover, $P(z)$ is analytic in z and the rank of $P(z)$ is constant. For every $z \in \tilde{\mathcal{R}} \setminus \Omega$, let

$$N(z) = H_\gamma(z)P(z) .$$

Then $N(z)$ is a nilpotent operator which is also analytic in z .

Our next goal is to show that the kernel of $H_\gamma(z)$ is one-dimensional for every $z \in \tilde{\mathcal{R}} \setminus \Omega$. In preparation of this, we settle a number of technical questions first.

3.3 Lemma. There exists a discrete subset $\Omega_0 \subset \tilde{\mathcal{R}}$ such that the following holds true: If $z \in \tilde{\mathcal{R}} \setminus \Omega_0$ and $\mathcal{L} \subset \ell^2(\mathbb{Z})$ is a finite-dimensional subspace which is invariant under $h(G(z))$, then \mathcal{L} contains a linear basis of eigenvectors of $h(G(z))$.

Proof: Let $\xi^{(z)}$ and ϑ_p be as in Theorem 2.16. Let

$$\Omega_0 = \{z \in \tilde{\mathcal{R}} \mid \text{There exists } \tilde{z} \in G^{-1}(\{p(z)\}) \text{ such that } \vartheta_0(\tilde{z}) = 0\} .$$

Since ϑ_0 is analytic, the set $M_0 = \{\tilde{z} \mid \vartheta_0(\tilde{z}) = 0\}$ is a discrete subset of $\tilde{\mathcal{R}}$. For every compact subset $K \subset \tilde{\mathcal{R}}$, the set $\{z \in K \mid z \in G^{-1}(\{p(\tilde{z})\}) \text{ for some } \tilde{z} \in M_0\}$ is finite. Hence Ω_0 is discrete.

Now let $z \in \tilde{\mathcal{R}} \setminus \Omega_0$ and consider the Jordan canonical form of $h(G(z))$ restricted on \mathcal{L} . We need to show that the nilpotent components in this decomposition are trivial. To this end, we need to show that if $\tilde{z} \in \tilde{\mathcal{R}}$ such that $p(\tilde{z})$ is an eigenvalue for $h(G(z))$ with

eigenvector $\xi^{(\tilde{z})}$, then there does not exist $\eta \in \ell^2(\mathbb{Z})$ such that $(h(G(z)) - p(\tilde{z}))\eta = \xi^{(\tilde{z})}$. Suppose that the opposite is true: There exists η with the said property. Then

$$\begin{aligned}\vartheta_0(z) &= \langle \xi^{(\tilde{z})}, \overline{\xi^{(\tilde{z})}} \rangle = \langle (h(G(z)) - p(\tilde{z}))\eta, \overline{\xi^{(\tilde{z})}} \rangle \\ &= \langle \eta, (h(G(z)) - p(\tilde{z}))^* \overline{\xi^{(\tilde{z})}} \rangle \\ &= \langle \eta, \overline{(h(G(z)) - p(\tilde{z}))\xi^{(\tilde{z})}} \rangle = 0 ,\end{aligned}$$

which means that $z \in \Omega_0$, thus contradicting our assumption on z . \blacksquare

We augment Ω , if necessary, to include the set Ω_0 .

3.4 Corollary. Let $z \in \tilde{\mathcal{R}} \setminus \Omega$ and for every $n \in \mathbb{Z}$, let t_n be chosen as in Proposition 3.2. Let

$$\mathcal{L} = \{ \xi \mid \xi_n = (t_n + t_n^{-1})^{-1} \eta_n , \text{ where } \eta \in \ell^2(\mathbb{Z}), H_\gamma(z)\eta = 0 \} .$$

Then \mathcal{L} contains a linear basis consisting of eigenvectors of $h_\gamma(G(z))$.

Proof: Since the kernel of $H_\gamma(z)$ is finite-dimensional, in view of Lemma 3.3 all that needs to be shown is that $h_\gamma(G(z))\mathcal{L} \subset \mathcal{L}$. Let $\eta \in \ell^2(\mathbb{Z})$, $H_\gamma(z)\eta = 0$ and let $\xi_n = (t_n + t_n^{-1})\eta$, $\tilde{\xi} = h_\gamma(G(z))\xi$, $\tilde{\eta}_n = (t_n + t_n^{-1})\tilde{\xi}_n$. Since $|t_n| |G_n(z)|^{-n}$ is constant,

$$\sum_{n=-\infty}^{\infty} |t_n + t_n^{-1}|^2 |\xi_n|^2 < \infty \text{ implies } \sum_{n=-\infty}^{\infty} |t_n + t_n^{-1}|^2 |\tilde{\xi}_n|^2 < \infty ,$$

which implies $\tilde{\eta} \in \ell^2(\mathbb{Z})$.

Next, since $D_{G(z)}k_\gamma D_{G(z)}$ and $h_\gamma(G(z))$ commute, and since $D_{G(z)}k_\gamma D_{G(z)}\xi = \Gamma(z)\xi$,

$$D_{G(z)}k_\gamma D_{G(z)}\tilde{\xi} = \Gamma(z)\tilde{\xi} ,$$

which in turn implies that $H_\gamma(z)\tilde{\eta} = 0$. In conclusion, $\tilde{\xi} = h_\gamma(G(z))\xi \in \mathcal{L}$. \blacksquare

Since N is analytic, the rank of $N(z)$ is constant on $\tilde{\mathcal{R}} \setminus \Omega$ with the possible exception of a discrete subset. We now enlarge Ω by this set of exceptional points.

3.5 Lemma. For every $z \in \tilde{\mathcal{R}} \setminus \Omega$, let $Q(z)$ be the projection whose kernel equals the range of $H_\gamma(z)$ and whose range equals the kernel of $H_\gamma(z)$. Then Q is analytic.

Proof: Let $z_0 \in \tilde{\mathcal{R}} \setminus \Omega$ and let $\mathcal{U} \subset \tilde{\mathcal{R}} \setminus \Omega$ be a simply connected open neighborhood of z_0 . We shall shrink \mathcal{U} when needed. By [Ko], II-§4.2 and VII-§1.3, there exists an analytic

function T from \mathcal{U} into $\mathcal{B}(\mathcal{H})$ such that $T(z)$ is invertible for every $z \in \mathcal{U}$ and

$$T(z)P(z_0)T(z)^{-1} = P(z) .$$

Let

$$\tilde{N}(z) = T^{-1}(z)N(z)T(z) .$$

Then \tilde{N} is analytic on \mathcal{U} and $\tilde{N}(z)$ is nilpotent.

We may consider $\tilde{N}(z)$ as an operator on the finite-dimensional subspace $\mathcal{L} = P(z_0)\ell^2(\mathbb{Z})$. We choose a linear basis in \mathcal{L} , and we denote the matrix representation of $\tilde{N}(z)$ with respect to this basis once again by $\tilde{N}(z)$. Let m be the dimension of \mathcal{L} and let \mathcal{B} be the canonical basis of \mathbb{C}^m . Then we choose a subset \mathcal{B}_1 of \mathcal{B} such that $\{\tilde{N}(z_0)e \mid e \in \mathcal{B}\}$ is a linear basis for $\tilde{N}(z_0)\mathbb{C}^m$. Since the functions $z \mapsto \tilde{N}(z)e$ are analytic in \mathcal{U} for all $e \in \mathcal{B}_1$, $\mathcal{B}_1(z) = \{\tilde{N}(z)e \mid e \in \mathcal{B}_1\}$ is a basis for $\tilde{N}(z)\mathbb{C}^m$ for every z in an open neighborhood of z_0 contained in \mathcal{U} . We replace \mathcal{U} by that neighborhood. Since \tilde{N} is nothing but a matrix whose entries are analytic functions, we can choose a submatrix M of \tilde{N} of type $r \times m$, where r equals the rank of $\tilde{N}(z_0)$, such that $M(z_0)$ has rank r . Again, since the entries of M are analytic in z , $r = \text{rank}(M(z)) = \text{rank}(\tilde{N}(z))$ for every z in an open neighborhood of z_0 contained in \mathcal{U} . Once again we replace \mathcal{U} by that neighborhood. We can now use Cramer's rule to solve a system of linear equations for analytic functions f_1, \dots, f_r on \mathcal{U} with values in \mathbb{C}^m such that $\mathcal{B}_2(z) = \{f_1(z), \dots, f_r(z)\}$ is a basis for the kernel of $\tilde{N}(z)$. Let $\tilde{Q}(z)$ be the projection whose range equals the kernel of $\tilde{N}(z)$ and whose kernel equals the range of $\tilde{N}(z)$. It follows that \tilde{Q} is analytic on \mathcal{U} . By construction $Q(z) = T(z)(\tilde{Q}(z) \circ P(z_0))T(z)^{-1}$. In conclusion, Q is locally analytic and hence analytic on $\tilde{\mathcal{R}} \setminus \Omega$ as claimed. ■

3.6 Lemma. Let F be a mapping from $\mathcal{R} \setminus p(\Omega)$ into the subsets of \mathcal{R} such that $F(z)$ contains exactly r elements. Suppose that F has the following properties:

- (I) If $z \in Sp(h(\delta))$, then $F(z) \subset Sp(h(\delta))$.
- (II) For every $z_0 \in \mathcal{R}$, there exists an open neighborhood \mathcal{U} of z_0 and for every j , $1 \leq j \leq r$, there exists an analytic function $s_j^{(z_0)}$ on \mathcal{U} such that

$$F(z) = \{s_1^{(z_0)}(z), \dots, s_r^{(z_0)}(z)\} \text{ for every } z \in \mathcal{U} .$$

Then $F(z) \subset \{z, -z\}$. (A mapping F having the property (II) is called an *analytic multifunction*).

Proof: For every $z \in \mathcal{R} \setminus p(\Omega)$, we form the polynomial

$$\left(X - s_1^{(z)}(z)\right) \cdot \dots \cdot \left(X - s_r^{(z)}(z)\right) = X^r + \sum_{j=0}^{r-1} f_j(z) X^j .$$

Then (II) entails that the coefficients f_j are analytic functions on $\mathcal{R} \setminus p(\Omega)$. By (I) the sets $F(z)$ are uniformly bounded in a punctured neighborhood of any point in $p(\Omega)$. It follows that the same is true for the functions f_j . Thus, all points in $p(\Omega)$ are removable singularities for f_j , and we may consider f_j as an analytic function on \mathcal{R} . It follows that the roots of the above polynomial are branches of analytic functions with exceptional point located in $p(\Omega)$ (see [Ko], II-§1.2).

Let d be the smallest number such that the logarithmic potential associated with the equilibrium distribution μ of $Sp(h)$ has a critical point y on the corresponding level curve. More precisely,

$$\int \log |y - t| d\mu(t) = d ,$$

while

$$\int (y - t)^{-1} d\mu(t) = 0 .$$

By Corollary 2.13, we know that there always exists a critical point. Let

$$\mathcal{R}_d = \left\{ z \in \mathbb{C} \mid \int \log |z - t| d\mu(t) > d \right\} .$$

Then there exists an analytic function G_d on \mathcal{R}_d such that

$$\log |G_d(z)| = \int \log |z - t| d\mu(t)$$

and

$$G_d(z) \in \mathbb{R}^+ \text{ whenever } z \in \mathbb{R}^+ .$$

Also, G_d has a simple pole at infinity. Let $\overline{\mathcal{R}}_d$ be the closure of \mathcal{R}_d and let M be the set of critical points in $\overline{\mathcal{R}}_d$. Then G_d has a continuous extension on $\overline{\mathcal{R}}_d \setminus M$ and it approaches two distinct points as z approaches a point in M . Let $z_0 \in \mathcal{R}_d \setminus p(\Omega)$. Then (I) entails that

$$\left. \begin{array}{l} s_j^{(z_0)}(z) \in \mathcal{R}_d , \\ \left| G_d \left(s_j^{(z_0)}(z) \right) \right| = |G_d(z)| \end{array} \right\} \text{ in a neighborhood } U \text{ of } z_0$$

for $1 \leq j \leq r$.

It follows that there exist $c_j \in \mathbb{T}$ such that

$$G_d \circ s_j^{(z_0)} = c_j G_d \text{ on } \mathcal{U}, \quad 1 \leq j \leq r.$$

Since this holds true for every $z_0 \in \mathcal{R}_d \setminus p(\Omega)$, (II) implies that the set of those scaling factors c_j is the same for all points in $\mathcal{R}_d \setminus p(\Omega)$. Hence

$$G_d \circ F(z) = \{c_1, \dots, c_r\} G_d(z),$$

for $z \in \mathcal{R}_d \setminus p(\Omega)$, where $G_d \circ F$ denotes the multifunction obtained by applying G_d to every element in $F(z)$. Since G_d is a conformal map from $\mathcal{R}_d \cup \{\infty\}$ onto $\{z \mid |z| > e^d\} \cup \{\infty\}$, it follows that there exist analytic functions s_1, \dots, s_r on \mathcal{R}_d such that

$$F(z) = \{s_1(z), \dots, s_r(z)\}, \quad z \in \mathcal{R}_d \setminus p(\Omega).$$

Every s_j is transformed via G_d into a rotation by c_j on $\{z \mid |z| > e^d\}$. Moreover, the analytic continuation of s_j across the boundary of $\overline{\mathcal{R}_d}$ does not result in the occurrence of exceptional points (branching-point) in $\overline{\mathcal{R}_d} \setminus \mathcal{R}_d$ or points of order larger than one. For, the existence of such points would conflict with the fact that all the critical points in M are simple. It follows that s_j has a differentiable extension on $\overline{\mathcal{R}_d}$ with non-vanishing derivatives. So, s_j has an inverse transformation with the same properties. Let \mathcal{K} be the group of transformations on $\overline{\mathcal{R}_d}$ which is generated by s_1, \dots, s_r and their inverses. Since the derivative $G'_d(z)$ approaches zero as z approaches a point in M , it follows that every $s \in \mathcal{K}$ maps M into M . Moreover, if $z \in M$ is a fixed point for some transformation in \mathcal{K} , that transformation must be the identity. Hence, since M is finite, \mathcal{K} is finite too. In particular, the group of scaling factors

$$\mathcal{C} = \{c \in \mathbb{T} \mid G_d \circ s = c G_d \text{ for some } s \in \mathcal{K}\}$$

is finite, which in turn means that $c^m = 1$ for every $c \in \mathcal{C}$, where m is the cardinality of \mathcal{C} . We want to show that $\mathcal{C} \subset \{-1, 1\}$.

To this end, we first note that $G_d(z) \subset G(p^{-1}(z))$, if we let $G_d(z)$ be the set of the two limit points of G_d at z in case z is in M . So, let $z \in M$. Since z is real, since $G \circ \sigma = \overline{G}$, and since $\sigma \circ \omega = \omega^{-1} \circ \sigma$, there exists a point $\tilde{z} \in p^{-1}(z)$ such that either

$$\sigma(\tilde{z}) = \tilde{z} \text{ or } \sigma(\tilde{z}) = \omega(z).$$

In either case, $G(p^{-1}(z)) \subset \{\lambda^n \mid n \in \mathbb{Z}\} \cup \{-\lambda^n \mid n \in \mathbb{Z}\}$. Now let $s \in \mathcal{K}$ and let $z' = s(z)$. Then

$$G_d(z), G_d(z') \subset \{\lambda^n \mid n \in \mathbb{Z}\} \cup \{-\lambda^n \mid n \in \mathbb{Z}\}.$$

Let $G_d \circ s = cG_d$ for some $c \in \mathcal{C}$. Since $c^m = 1$ on the one hand, while λ is non-periodic on the other hand, $c \in \{-1, 1\}$ as claimed. Finally, since $-z \in \mathcal{R}$ whenever $z \in \mathcal{R}$, we conclude that $r \leq 2$ and $F(z) \subset \{-z, z\}$ for every $z \in \mathcal{R}$, by virtue of analytic continuation. ■

3.7 Proposition. The kernel of $H_\gamma(z)$ is one-dimensional for every $z \in \tilde{\mathcal{R}} \setminus \Omega$.

Proof: For every $z \in \tilde{\mathcal{R}} \setminus \Omega$ choose $t_n(z)$ as in Proposition 3.2, that is,

$$H_\gamma(z)\eta = 0 \text{ if and only if } k_\gamma D_{G(z)}\xi = \Gamma(z)D_{G(z)^{-1}}\xi ,$$

whenever $\eta \in \ell^2(\mathbb{Z})$ and $\xi_n = (t_n(z) + t_n(z)^{-1})^{-1}\eta_n$.

Let $\tilde{h}_\gamma(G(z))$ be defined as follows:

$$\left(\tilde{h}_\gamma(G(z))\eta\right)_n = s_n(z)\gamma\eta_{n+1} + s_n(z)^{-1}\gamma^{-1}\eta_{n-1} + \beta(G(z)\lambda^n + G(z)^{-1}\lambda^{-n})\eta_n ,$$

where

$$s_n(z) = (t_{n+1}(z) + t_{n+1}(z)^{-1}) (t_n(z) + t_n(z)^{-1})^{-1} .$$

Then $\tilde{h}_\gamma(G(z))$ is a bounded operator. Since Γ is analytic, $\tilde{h}_\gamma(G(z))$ is analytic in z . Let

$$B(z) = Q(z)\tilde{h}_\gamma(G(z))Q(z) , \quad z \in \tilde{\mathcal{R}} \setminus \Omega .$$

Then $B(z)$ is a finite rank operator and B is analytic by Lemma 3.5. By Corollary 3.4 the cardinality of the spectrum of $B(z)$ equals the rank of $B(z)$. The rank of $B(z)$, however, equals the dimension of the kernel of $H_\gamma(z)$. So, in order to settle the claim it has to be shown that the spectrum of $B(z)$ contains exactly one point. Since $Sp(B(\omega(z))) = Sp(B(z))$, we can define a multi-function F on $\mathcal{R} \setminus p(\Omega)$,

$$F(z) = Sp(B(\tilde{z})) , \quad p(\tilde{z}) = z .$$

Then F enjoys the properties stated in Lemma 3.6. It follows that $F(z) \subset \{-z, z\}$ for every $z \in \mathcal{R} \setminus p(\Omega)$. Since $F(p(\tilde{z})) \subset Sp(h_\gamma(G(\tilde{z})))$ for every $\tilde{z} \in \tilde{\mathcal{R}} \setminus \Omega$, and since $-z$ cannot be an eigenvalue of $h_\gamma(G(\tilde{z}))$ if z is one, we conclude that $Sp(B(\tilde{z}))$ contains exactly one element, as claimed. (Compare this with the remark following Proposition 3.2.) ■

As an immediate consequence of Proposition 3.7, we obtain the following corollary, announced in the introductory remarks to this paragraph:

3.8 Corollary. If $z_1, z_2 \in \widetilde{\mathcal{R}} \setminus \Omega$ have the property that $(G(z_1), \Gamma(z_1)) = (G(z_2), \Gamma(z_2))$, then $z_1 = z_2$.

4. The case $|\delta| = 1$

In this final paragraph we will describe how certain features from the preceding two paragraphs carry over to the operator $h = h(1)$.

First, h is a fixed point of the automorphism ρ_β , and therefore the operators h and k commute. This information can be used to show the following:

$$(4.1) \quad \text{The spectrum of } h_\gamma = h_\gamma(1) \text{ is constant for } \beta^{-1} \leq |\gamma| \leq \beta .$$

Moreover, if ξ is a solution of

$$h(x)\xi = \chi\xi ; \quad \chi \in Sp(h) , \quad x \in \mathbb{T} ,$$

and $|\xi_n|$ grows moderately as $|n| \rightarrow \infty$, then

$$D_x k D_x \xi = c \xi$$

for some complex number c . Hence

$$D_{\bar{x}} k^{-1} D_{\bar{x}} \bar{\xi} = \overline{D_x k D_x \xi} = \bar{c} \bar{\xi} ,$$

or

$$D_x k D_x \bar{\xi} = \bar{c}^{-1} \bar{\xi} .$$

So, if ξ is real, then $|c| = 1$. This information, in conjunction with (4.1), can be used to show the following:

$$(4.2) \quad Sp(D_x k_\gamma D_x) = \mathbb{T} \text{ for } \beta^{-1} \leq |\gamma| \leq \beta \text{ and } x \in \mathbb{T} .$$

In analogy to the operators $H_\gamma(z)$ considered in paragraph 3, we define for $\gamma \in \mathbb{R}$ close to 1 and $\theta, \nu \in \mathbb{R}$;

$$(H_\gamma(\theta, \nu)\eta)_n = \sum_{j=-\infty}^{\infty} a_j \eta_{n+j} + \tan \pi(\alpha n^2 + 2\theta n + \nu) \eta_n ,$$

where $|\eta_n|$ grows moderately as $|n| \rightarrow \infty$, and

$$\tan g(\gamma u) = \sum_{j=-\infty}^{\infty} a_j u^j ,$$

with g as in (1.6). Then $H_\gamma(\theta, \nu)$ determines an unbounded operator on $\ell^2(\mathbb{Z})$ with a dense domain, as long as

$$(\theta, \nu) \notin \Omega_0 = \left\{ (\tilde{\theta}, \tilde{\nu}) \mid \alpha n^2 + 2\tilde{\theta}n + \tilde{\nu} \in \frac{1}{2}\mathbb{Z} \right\} .$$

As $\Gamma(z)$ and $G(z)$ approach suitable complex numbers of modulus one, respectively, $H_\gamma(z)$ is seen to approach an operator of this type, on the linear subspace of all vectors with finitely many non-vanishing components only, say.

As in Proposition 3.2, one can now show the following: Let $\xi \in \ell^2(\mathbb{Z})$ be an eigenvector of $h_\gamma(x)$ and suppose that

$$D_x k_\gamma D_x \xi = c\xi .$$

Let

$$\eta_n = \cos \pi(\alpha n^2 + 2\theta n + \nu) \xi_n ,$$

where $e^{2\pi\theta i} = x$, $e^{2\pi\nu i} = c$. Then

$$H_\gamma(\theta, \nu)\eta = 0 ,$$

in case $(\theta, \nu) \notin \Omega_0$. Thus, information about the eigenvalue problem for the operator $h_\gamma(\theta)$ can be transformed into information regarding the kernel of an operator of the form $H_\gamma(\theta, \nu)$, and to some degree, this works in the opposite direction as well.

If β is sufficiently large, then we may choose $\gamma = 1$. In this case, $H(\theta, \nu) = H_1(\theta, \nu)$ becomes an essentially self-adjoint operator. A complete analysis of the spectral properties of operators of this type can be obtained in case $\alpha = 0$ and θ is an irrational number satisfying a stronger diophantine condition than the one that has been assumed to be valid for α in this paper (cf. [PF], Chapter VII, §18).

Appendix

Two items related to a system of difference equations which have been used explicitly in the text will be assembled. For more details, we refer to [R1] and [R2].

A1. Consider the system

$$(A1.1) \quad \begin{cases} \cos(\pi\alpha q + \theta)(X_{p-1,q} + X_{p+1,q}) + \beta \cos(\pi\alpha p + \theta)(X_{p,q-1} + X_{p,q+1}) = zX_{pq} \\ \sin(\pi\alpha q + \theta)(X_{p-1,q} - X_{p+1,q}) - \beta \sin(\pi\alpha p + \theta)(X_{p,q-1} - X_{p,q+1}) = 0 . \end{cases}$$

As in [R1], one can construct a recursion

$$\begin{pmatrix} X_{p+2,p+2} \\ X_{p+2,p+1} \\ X_{p+1,p+1} \end{pmatrix} = F_p \begin{pmatrix} X_{p+1,p+1} \\ X_{p+1,p} \\ X_{pp} \end{pmatrix} , \quad p \geq 0 ,$$

where $\{X_{pq}\}$ is any solution of (A1.1) and

$$F_p = E_p D_p C_p ,$$

$$C_p = \begin{bmatrix} -\frac{\beta \sin[2\pi\alpha(p+1) + 2\theta]}{\sin \pi\alpha} & \frac{\chi \sin[\pi\alpha(p+1) + \theta]}{\sin \pi\alpha} & -\frac{\sin[\pi\alpha(2p+1) + 2\theta]}{\sin \pi\alpha} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$D_p = \begin{bmatrix} 0 & \frac{\chi}{2 \cos[\pi\alpha(p+1) + \theta]} & -\beta \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$E_p = \begin{bmatrix} -\frac{\chi \sin[\pi\alpha(p+1) + \theta]}{\beta \sin[\pi\alpha(2p+3) + 2\theta]} & -\frac{\sin[2\pi\alpha(p+1) + 2\theta]}{\beta \sin[\pi\alpha(2p+3) + 2\theta]} & \frac{\sin \pi\alpha}{\sin[\pi\alpha(2p+3) + 2\theta]} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} .$$

If $\{X_{pq}\}$ and $\{Y_{pq}\}$ are solutions of (A1.1), then

$$X_{p+1,p+1}Y_{p+2,p+2} - X_{p+2,p+2}Y_{p+1,p+1} = -\beta \frac{\sin(\pi\alpha + \theta)}{\sin[\pi\alpha(2p+3) + 2\theta]} \cdot (X_{00}Y_{11} - X_{11}Y_{00}) .$$

Now suppose $\theta \notin \pi\alpha\mathbb{Z}$. Then the following holds true: If $(X_{00}, X_{11}) \neq (0, 0)$, $(Y_{00}, Y_{11}) \neq (0, 0)$ and the sequence

$$\{X_{pp}Y_{p+1,p+1} - X_{p+1,p+1}Y_{pp}\}_{p \geq 0}$$

is bounded, while $\{\sup_{p \in \mathbb{Z}} |X_{pq}|\}_{q \geq 0}$ and $\{\sup_{p \in \mathbb{Z}} |Y_{pq}|\}_{q \leq 0}$ are bounded as well, then $\{X_{pq}\}$ and $\{Y_{pq}\}$ must be linearly dependent.

A2. Now consider the system (A1.1) for $\theta = 0$. Let

$$(h - z)^{-1} = \sum_{p, q \in \mathbb{Z}} c_{pq}(z) w_{pq}, \quad z \in \mathcal{R} = \mathbb{C} \setminus Sp(h).$$

Then c_{pq} is analytic and $\{c_{pq}(z)\}$ solves the system (A1.1), except for $p = q = 0$.

By [R2], Paragraph 4, there exist polynomials d_{pq} which are either zero or of degree $||p| - |q|| - 1$ such that $\{d_{pq}(z)\}$ solves the system (A1.1) except for $p = q = 0$ and

$$d_{pq}(z) = 0 \text{ for } q \geq -|p|; \quad d_{p, -p-1}(z) = (-1)^p \beta^{-p-1} \text{ for } p \geq 0.$$

If $\{X_{pq}\}$ is a solution of (A1.1) for some $z \in \mathcal{R}$ which decays exponentially uniformly in p as $q \rightarrow \infty$, then $\{X_{pq}\}$ and $\{c_{pq}(z) - d_{pq}(z)\}$ must be linearly independent.

References

- [AS] L. V. Ahlfors, L. Sario, “Riemann Surfaces”, Princeton Mathematical Series, 1976.
- [Ko] T. Kato, “Perturbation Theory for Linear Operators” Second Edition, Springer-Verlag 1976.
- [Kn] Y. Katznelson, “An Introduction to Harmonic Analysis”, Second Corrected Edition, Dover 1976.
- [PF] L. Pastur, A. Figotin, “Spectra of Random and Almost-Periodic Operators”, Springer-Verlag 1992.
- [R1] N. Riedel, “Almost Mathieu operators and rotation C^* -algebras”, Proc. London Math. Soc. 3 (56), (1988), 281–302.
- [R2] N. Riedel, “The spectrum of a class of almost periodic operators”, submitted since July 1992.
- [R3] N. Riedel, “Regularity of the spectrum for the almost Mathieu operator” (September 1992), Proc. Amer. Math. Soc., to appear.

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